Chapter 4

Motion in Two Dimensions

▲ *Lava spews from a volcanic eruption. Notice the parabolic paths of embers projected into the air. We will find in this chapter that all projectiles follow a parabolic path in the absence of air resistance. (© Arndt/Premium Stock/PictureQuest)*

CHAPTER OUTLINE

In this chapter we explore the kinematics of a particle moving in two dimensions. Knowing the basics of two-dimensional motion will allow us to examine—in future chapters—a wide variety of motions, ranging from the motion of satellites in orbit to the motion of electrons in a uniform electric field. We begin by studying in greater detail the vector nature of position, velocity, and acceleration. As in the case of one-dimensional motion, we derive the kinematic equations for two-dimensional motion from the fundamental definitions of these three quantities. We then treat projectile motion and uniform circular motion as special cases of motion in two dimensions. We also discuss the concept of relative motion, which shows why observers in different frames of reference may measure different positions, velocities, and accelerations for a given particle.

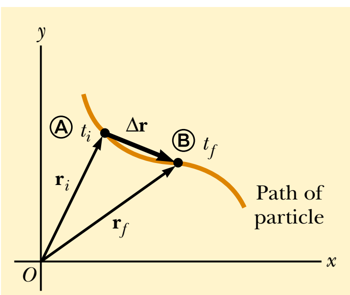


Figure 4.1 A particle moving in the *xy* plane is located with the position vector **r** drawn from the origin to the particle. The displacement of the particle as it moves from Ⓐ to Ⓑ in the time interval $\Delta t = t_f - t_i$ is equal to the vector $\Delta\mathbf{r} = \mathbf{r}_f - \mathbf{r}_i$.

4.1 The Position, Velocity, and Acceleration Vectors

In Chapter 2 we found that the motion of a particle moving along a straight line is completely known if its position is known as a function of time. Now let us extend this idea to motion in the *xy* plane. We begin by describing the position of a particle by its **position vector r**, drawn from the origin of some coordinate system to the particle located in the *xy* plane, as in Figure 4.1. At time t_i the particle is at point Ⓐ, described by position vector $\mathbf{r}_i$. At some later time t_f it is at point Ⓑ, described by position vector $\mathbf{r}_f$. The path from Ⓐ to Ⓑ is not necessarily a straight line. As the particle moves from Ⓐ to Ⓑ in the time interval $\Delta t = t_f - t_i$, its position vector changes from $\mathbf{r}_i$ to $\mathbf{r}_f$. As we learned in Chapter 2, displacement is a vector, and the displacement of the particle is the difference between its final position and its initial position. We now define the **displacement vector** $\Delta\mathbf{r}$ for the particle of Figure 4.1 as being the difference between its final position vector and its initial position vector:

Displacement vector

$$\Delta\mathbf{r} \equiv \mathbf{r}_f - \mathbf{r}_i \tag{4.1}$$

The direction of $\Delta\mathbf{r}$ is indicated in Figure 4.1. As we see from the figure, the magnitude of $\Delta\mathbf{r}$ is *less* than the distance traveled along the curved path followed by the particle.

As we saw in Chapter 2, it is often useful to quantify motion by looking at the ratio of a displacement divided by the time interval during which that displacement occurs, which gives the rate of change of position. In two-dimensional (or three-dimensional) kinematics, everything is the same as in one-dimensional kinematics except that we must now use full vector notation rather than positive and negative signs to indicate the direction of motion.

We define the **average velocity** of a particle during the time interval Δt as the displacement of the particle divided by the time interval:

Average velocity

$$\overline{\mathbf{v}} \equiv \frac{\Delta\mathbf{r}}{\Delta t} \tag{4.2}$$

Mark C. Burnett/Photo Researchers, Inc.

Figure 4.2 Bird's-eye view of a baseball diamond. A batter who hits a home run travels around the bases, ending up where he began. Thus, his average velocity for the entire trip is zero. His average speed, however, is not zero and is equal to the distance around the bases divided by the time interval during which he runs around the bases.

Multiplying or dividing a vector quantity by a positive scalar quantity such as Δt changes only the magnitude of the vector, not its direction. Because displacement is a vector quantity and the time interval is a positive scalar quantity, we conclude that the average velocity is a vector quantity directed along $\Delta\mathbf{r}$.

Note that the average velocity between points is *independent of the path* taken. This is because average velocity is proportional to displacement, which depends only on the initial and final position vectors and not on the path taken. As with one-dimensional motion, we conclude that if a particle starts its motion at some point and returns to this point via any path, its average velocity is zero for this trip because its displacement is zero. Figure 4.2 suggests such a situation in a baseball park. When a batter hits a home run, he runs around the bases and returns to home plate. Thus, his average velocity is zero during this trip. His average speed, however, is not zero.

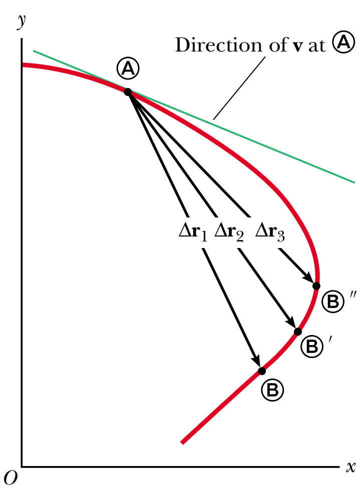


Figure 4.3 As a particle moves between two points, its average velocity is in the direction of the displacement vector $\Delta\mathbf{r}$. As the end point of the path is moved from Ⓑ to Ⓑ′ to Ⓑ″, the respective displacements and corresponding time intervals become smaller and smaller. In the limit that the end point approaches Ⓐ, Δt approaches zero, and the direction of $\Delta\mathbf{r}$ approaches that of the line tangent to the curve at Ⓐ. By definition, the instantaneous velocity at Ⓐ is directed along this tangent line.

Consider again the motion of a particle between two points in the xy plane, as shown in Figure 4.3. As the time interval over which we observe the motion becomes smaller and smaller, the direction of the displacement approaches that of the line tangent to the path at Ⓐ. The **instantaneous velocity v** is defined as the limit of the average velocity $\Delta\mathbf{r}/\Delta t$ as Δt approaches zero:

$$\mathbf{v} \equiv \lim_{\Delta t \to 0} \frac{\Delta\mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt} \tag{4.3}$$

That is, the instantaneous velocity equals the derivative of the position vector with respect to time. The direction of the instantaneous velocity vector at any point in a particle's path is along a line tangent to the path at that point and in the direction of motion.

The magnitude of the instantaneous velocity vector $v = |\mathbf{v}|$ is called the *speed,* which is a scalar quantity.

As a particle moves from one point to another along some path, its instantaneous velocity vector changes from $\mathbf{v}_i$ at time t_i to $\mathbf{v}_f$ at time t_f. Knowing the velocity at these points allows us to determine the average acceleration of the particle—the **average acceleration $\overline{\mathbf{a}}$** of a particle as it moves is defined as the change in the instantaneous velocity vector $\Delta\mathbf{v}$ divided by the time interval Δt during which that change occurs:

$$\overline{\mathbf{a}} \equiv \frac{\mathbf{v}_f - \mathbf{v}_i}{t_f - t_i} = \frac{\Delta\mathbf{v}}{\Delta t} \tag{4.4}$$

PITFALL PREVENTION

4.1 Vector Addition

While the vector addition discussed in Chapter 3 involves *displacement* vectors, vector addition can be applied to *any* type of vector quantity. Figure 4.4, for example, shows the addition of *velocity* vectors using the graphical approach.

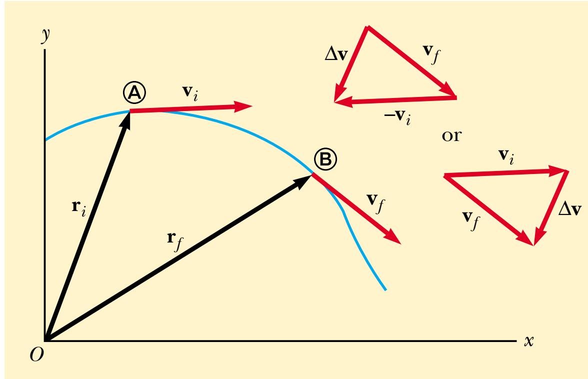


Figure 4.4 A particle moves from position Ⓐ to position Ⓑ. Its velocity vector changes from $\mathbf{v}_i$ to $\mathbf{v}_f$. The vector diagrams at the upper right show two ways of determining the vector $\Delta\mathbf{v}$ from the initial and final velocities.

Because $\bar{\mathbf{a}}$ is the ratio of a vector quantity $\Delta\mathbf{v}$ and a positive scalar quantity Δt, we conclude that average acceleration is a vector quantity directed along $\Delta\mathbf{v}$. As indicated in Figure 4.4, the direction of $\Delta\mathbf{v}$ is found by adding the vector $-\mathbf{v}_i$ (the negative of $\mathbf{v}_i$) to the vector $\mathbf{v}_f$, because by definition $\Delta\mathbf{v} = \mathbf{v}_f - \mathbf{v}_i$.

When the average acceleration of a particle changes during different time intervals, it is useful to define its instantaneous acceleration. The **instantaneous acceleration a** is defined as the limiting value of the ratio $\Delta\mathbf{v}/\Delta t$ as Δt approaches zero:

Instantaneous acceleration

$$\mathbf{a} \equiv \lim_{\Delta t \to 0} \frac{\Delta\mathbf{v}}{\Delta t} = \frac{d\mathbf{v}}{dt} \tag{4.5}$$

In other words, the instantaneous acceleration equals the derivative of the velocity vector with respect to time.

It is important to recognize that various changes can occur when a particle accelerates. First, the magnitude of the velocity vector (the speed) may change with time as in straight-line (one-dimensional) motion. Second, the direction of the velocity vector may change with time even if its magnitude (speed) remains constant, as in curved-path (two-dimensional) motion. Finally, both the magnitude and the direction of the velocity vector may change simultaneously.

Quick Quiz 4.1 Which of the following cannot *possibly* be accelerating? (a) An object moving with a constant speed (b) An object moving with a constant velocity (c) An object moving along a curve.

Quick Quiz 4.2 Consider the following controls in an automobile: gas pedal, brake, steering wheel. The controls in this list that cause an acceleration of the car are (a) all three controls (b) the gas pedal and the brake (c) only the brake (d) only the gas pedal.

4.2 Two-Dimensional Motion with Constant Acceleration

In Section 2.5, we investigated one-dimensional motion in which the acceleration is constant because this type of motion is common. Let us consider now two-dimensional motion during which the acceleration remains constant in both magnitude and direction. This will also be useful for analyzing some common types of motion.

The position vector for a particle moving in the xy plane can be written

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} \tag{4.6}$$

where x, y, and $\mathbf{r}$ change with time as the particle moves while the unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ remain constant. If the position vector is known, the velocity of the particle can be obtained from Equations 4.3 and 4.6, which give

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} \tag{4.7}$$

Because $\mathbf{a}$ is assumed constant, its components a_x and a_y also are constants. Therefore, we can apply the equations of kinematics to the x and y components of the velocity vector. Substituting, from Equation 2.9, $v_{xf} = v_{xi} + a_x t$ and $v_{yf} = v_{yi} + a_y t$ into Equation 4.7 to determine the final velocity at any time t, we obtain

$$\begin{aligned}\mathbf{v}_f &= (v_{xi} + a_x t)\hat{\mathbf{i}} + (v_{yi} + a_y t)\hat{\mathbf{j}}\\ &= (v_{xi}\hat{\mathbf{i}} + v_{yi}\hat{\mathbf{j}}) + (a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}})t\end{aligned}$$

$$\mathbf{v}_f = \mathbf{v}_i + \mathbf{a}t \tag{4.8}$$

Velocity vector as a function of time

This result states that the velocity of a particle at some time t equals the vector sum of its initial velocity $\mathbf{v}_i$ and the additional velocity $\mathbf{a}t$ acquired at time t as a result of constant acceleration. It is the vector version of Equation 2.9.

Similarly, from Equation 2.12 we know that the x and y coordinates of a particle moving with constant acceleration are

$$x_f = x_i + v_{xi}t + \tfrac{1}{2}a_x t^2 \qquad y_f = y_i + v_{yi}t + \tfrac{1}{2}a_y t^2$$

Substituting these expressions into Equation 4.6 (and labeling the final position vector $\mathbf{r}_f$) gives

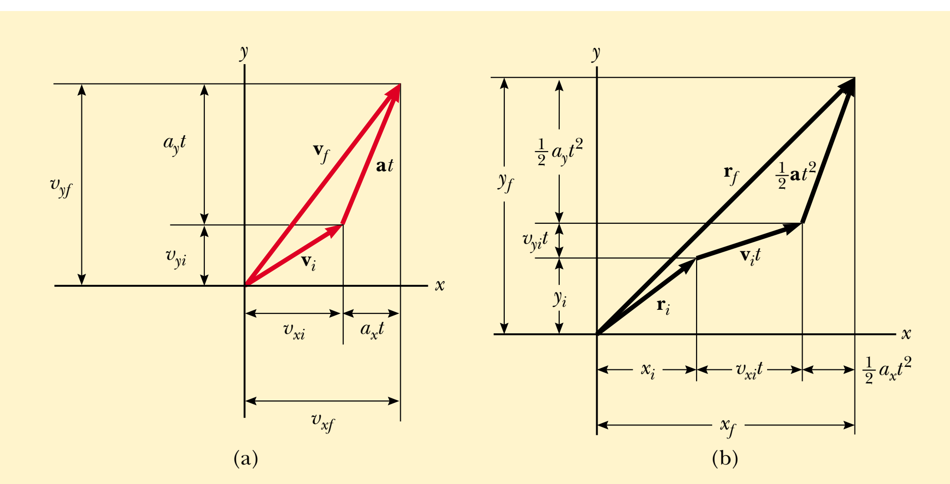
$$\begin{aligned}\mathbf{r}_f &= (x_i + v_{xi}t + \tfrac{1}{2}a_x t^2)\hat{\mathbf{i}} + (y_i + v_{yi}t + \tfrac{1}{2}a_y t^2)\hat{\mathbf{j}}\\ &= (x_i\hat{\mathbf{i}} + y_i\hat{\mathbf{j}}) + (v_{xi}\hat{\mathbf{i}} + v_{yi}\hat{\mathbf{j}})t + \tfrac{1}{2}(a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}})t^2\end{aligned}$$

$$\mathbf{r}_f = \mathbf{r}_i + \mathbf{v}_i t + \tfrac{1}{2}\mathbf{a}t^2 \tag{4.9}$$

Position vector as a function of time

which is the vector version of Equation 2.12. This equation tells us that the position vector $\mathbf{r}_f$ is the vector sum of the original position $\mathbf{r}_i$, a displacement $\mathbf{v}_i t$ arising from the initial velocity of the particle and a displacement $\frac{1}{2}\mathbf{a}t^2$ resulting from the constant acceleration of the particle.

Graphical representations of Equations 4.8 and 4.9 are shown in Figure 4.5. Note from Figure 4.5a that $\mathbf{v}_f$ is generally not along the direction of either $\mathbf{v}_i$ or $\mathbf{a}$ because the relationship between these quantities is a vector expression. For the same reason,



Active Figure 4.5 Vector representations and components of (a) the velocity and (b) the position of a particle moving with a constant acceleration $\mathbf{a}$.

At the Active Figures link at http://www.pse6.com, you can investigate the effect of different initial positions and velocities on the final position and velocity (for constant acceleration).

from Figure 4.5b we see that $\mathbf{r}_f$ is generally not along the direction of $\mathbf{v}_i$ or $\mathbf{a}$. Finally, note that $\mathbf{v}_f$ and $\mathbf{r}_f$ are generally not in the same direction.

Because Equations 4.8 and 4.9 are vector expressions, we may write them in component form:

$$\mathbf{v}_f = \mathbf{v}_i + \mathbf{a}t \quad \begin{cases} v_{xf} = v_{xi} + a_x t \\ v_{yf} = v_{yi} + a_y t \end{cases} \tag{4.8a}$$

$$\mathbf{r}_f = \mathbf{r}_i + \mathbf{v}_i t + \tfrac{1}{2}\mathbf{a}t^2 \quad \begin{cases} x_f = x_i + v_{xi}t + \frac{1}{2}a_x t^2 \\ y_f = y_i + v_{yi}t + \frac{1}{2}a_y t^2 \end{cases} \tag{4.9a}$$

These components are illustrated in Figure 4.5. The component form of the equations for $\mathbf{v}_f$ and $\mathbf{r}_f$ show us that two-dimensional motion at constant acceleration is equivalent to two *independent* motions—one in the x direction and one in the y direction—having constant accelerations a_x and a_y.

Example 4.1 Motion in a Plane

A particle starts from the origin at $t = 0$ with an initial velocity having an x component of 20 m/s and a y component of -15 m/s. The particle moves in the xy plane with an x component of acceleration only, given by $a_x = 4.0\ \text{m/s}^2$.

(A) Determine the components of the velocity vector at any time and the total velocity vector at any time.

Solution After carefully reading the problem, we *conceptualize* what is happening to the particle. The components of the initial velocity tell us that the particle starts by moving toward the right and downward. The x component of velocity starts at 20 m/s and increases by 4.0 m/s every second. The y component of velocity never changes from its initial value of -15 m/s. We sketch a rough motion diagram of the situation in Figure 4.6. Because the particle is accelerating in the $+x$ direction, its velocity component in this direction will increase, so that the path will curve as shown in the diagram. Note that the spacing between successive images increases as time goes on because the speed is increasing. The placement of the acceleration and velocity vectors in Figure 4.6 helps us to further conceptualize the situation.

Because the acceleration is constant, we *categorize* this problem as one involving a particle moving in two dimensions with constant acceleration. To *analyze* such a problem, we use the equations developed in this section. To begin the mathematical analysis, we set $v_{xi} = 20$ m/s, $v_{yi} = -15$ m/s, $a_x = 4.0\ \text{m/s}^2$, and $a_y = 0$.

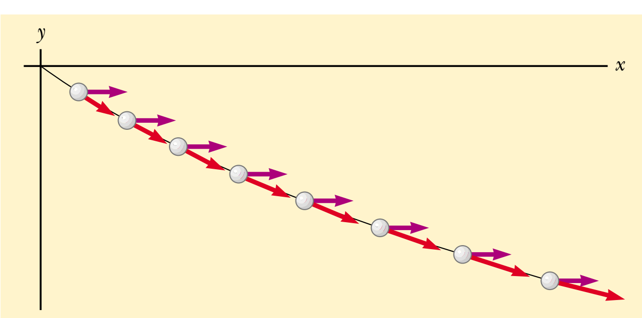


Figure 4.6 (Example 4.1) Motion diagram for the particle.

Equations 4.8a give

(1) $v_{xf} = v_{xi} + a_x t = (20 + 4.0t)$ m/s

(2) $v_{yf} = v_{yi} + a_y t = -15\ \text{m/s} + 0 = -15$ m/s

Therefore

$$\mathbf{v}_f = v_{xi}\hat{\mathbf{i}} + v_{yi}\hat{\mathbf{j}} = \boxed{[(20 + 4.0t)\hat{\mathbf{i}} - 15\hat{\mathbf{j}}]\ \text{m/s}}$$

We could also obtain this result using Equation 4.8 directly, noting that $\mathbf{a} = 4.0\hat{\mathbf{i}}\ \text{m/s}^2$ and $\mathbf{v}_i = [20\hat{\mathbf{i}} - 15\hat{\mathbf{j}}]$ m/s. To *finalize* this part, notice that the x component of velocity increases in time while the y component remains constant; this is consistent with what we predicted.

(B) Calculate the velocity and speed of the particle at $t = 5.0$ s.

Solution With $t = 5.0$ s, the result from part (A) gives

$$\mathbf{v}_f = [(20 + 4.0(5.0))\hat{\mathbf{i}} - 15\hat{\mathbf{j}}]\ \text{m/s} = \boxed{(40\hat{\mathbf{i}} - 15\hat{\mathbf{j}})\ \text{m/s}}$$

This result tells us that at $t = 5.0$ s, $v_{xf} = 40$ m/s and $v_{yf} = -15$ m/s. Knowing these two components for this two-dimensional motion, we can find both the direction and the magnitude of the velocity vector. To determine the angle θ that $\mathbf{v}$ makes with the x axis at $t = 5.0$ s, we use the fact that $\tan\theta = v_{yf}/v_{xf}$:

$$(3) \qquad \theta = \tan^{-1}\left(\frac{v_{yf}}{v_{xf}}\right) = \tan^{-1}\left(\frac{-15\ \text{m/s}}{40\ \text{m/s}}\right) = \boxed{-21°}$$

where the negative sign indicates an angle of 21° below the positive x axis. The speed is the magnitude of $\mathbf{v}_f$:

$$v_f = |\mathbf{v}_f| = \sqrt{v_{xf}^2 + v_{yf}^2} = \sqrt{(40)^2 + (-15)^2}\ \text{m/s} = \boxed{43\ \text{m/s}}$$

To *finalize* this part, we notice that if we calculate v_i from the x and y components of $\mathbf{v}_i$, we find that $v_f > v_i$. Is this consistent with our prediction?

(C) Determine the x and y coordinates of the particle at any time t and the position vector at this time.

Solution Because $x_i = y_i = 0$ at $t = 0$, Equation 4.9a gives

$$x_f = v_{xi}t + \tfrac{1}{2}a_x t^2 = \boxed{(20t + 2.0t^2)\text{ m}}$$

$$y_f = v_{yi}t = \boxed{(-15t)\text{ m}}$$

Therefore, the position vector at any time t is

$$(4) \qquad \mathbf{r}_f = x_f\hat{\mathbf{i}} + y_f\hat{\mathbf{j}} = \boxed{[(20t + 2.0t^2)\hat{\mathbf{i}} - 15t\hat{\mathbf{j}}]\text{ m}}$$

(Alternatively, we could obtain $\mathbf{r}_f$ by applying Equation 4.9 directly, with $\mathbf{v}_f = (20\hat{\mathbf{i}} - 15\hat{\mathbf{j}})$ m/s and $\mathbf{a} = 4.0\hat{\mathbf{i}}$ m/s^2. Try it!) Thus, for example, at $t = 5.0$ s, $x = 150$ m, $y = -75$ m, and $\mathbf{r}_f = (150\hat{\mathbf{i}} - 75\hat{\mathbf{j}})$ m. The magnitude of the displacement of the particle from the origin at $t = 5.0$ s is the magnitude of $\mathbf{r}_f$ at this time:

$$r_f = |\mathbf{r}_f| = \sqrt{(150)^2 + (-75)^2}\text{m} = 170\text{m}$$

Note that this is *not* the distance that the particle travels in this time! Can you determine this distance from the available data?

To *finalize* this problem, let us consider a limiting case for very large values of t in the following **What If?**

What If? **What if we wait a very long time and then observe the motion of the particle? How would we describe the motion of the particle for large values of the time?**

Answer Looking at Figure 4.6, we see the path of the particle curving toward the x axis. There is no reason to assume that this tendency will change, so this suggests that the path will become more and more parallel to the x axis as time grows large. Mathematically, let us consider Equations (1) and (2). These show that the y component of the velocity remains constant while the x component grows linearly with t. Thus, when t is very large, the x component of the velocity will be much larger than the y component, suggesting that the velocity vector becomes more and more parallel to the x axis.

Equation (3) gives the angle that the velocity vector makes with the x axis. Notice that $\theta \to 0$ as the denominator (v_{xf}) becomes much larger than the numerator (v_{yf}).

Despite the fact that the velocity vector becomes more and more parallel to the x axis, the particle does not approach a limiting value of y. Equation (4) shows that both x_f and y_f continue to grow with time, although x_f grows much faster.

4.3 Projectile Motion

Assumptions of projectile motion

Anyone who has observed a baseball in motion has observed projectile motion. The ball moves in a curved path, and its motion is simple to analyze if we make two assumptions: (1) the free-fall acceleration $\mathbf{g}$ is constant over the range of motion and is directed downward,[1] and (2) the effect of air resistance is negligible.[2] With these assumptions, we find that the path of a projectile, which we call its *trajectory,* is *always* a parabola. **We use these assumptions throughout this chapter.**

To show that the trajectory of a projectile is a parabola, let us choose our reference frame such that the y direction is vertical and positive is upward. Because air resistance is neglected, we know that $a_y = -g$ (as in one-dimensional free fall) and that $a_x = 0$. Furthermore, let us assume that at $t = 0$, the projectile leaves the origin ($x_i = y_i = 0$) with speed v_i, as shown in Figure 4.7. The vector $\mathbf{v}_i$ makes an angle θ_i with the horizontal. From the definitions of the cosine and sine functions we have

$$\cos\theta_i = v_{xi}/v_i \qquad \sin\theta_i = v_{yi}/v_i$$

Therefore, the initial x and y components of velocity are

$$v_{xi} = v_i\cos\theta_i \qquad v_{yi} = v_i\sin\theta_i \tag{4.10}$$

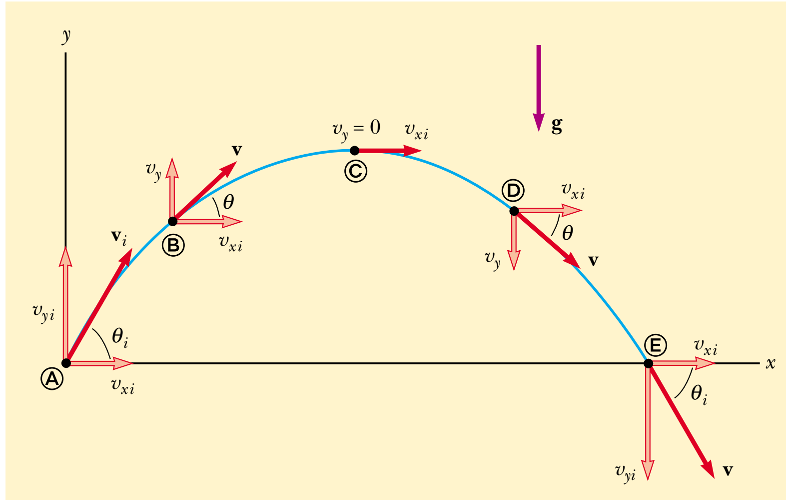
Substituting the x component into Equation 4.9a with $x_i = 0$ and $a_x = 0$, we find that

$$x_f = v_{xi}t = (v_i\cos\theta_i)t \tag{4.11}$$

[1] This assumption is reasonable as long as the range of motion is small compared with the radius of the Earth (6.4×10^6 m). In effect, this assumption is equivalent to assuming that the Earth is flat over the range of motion considered.

[2] This assumption is generally *not* justified, especially at high velocities. In addition, any spin imparted to a projectile, such as that applied when a pitcher throws a curve ball, can give rise to some very interesting effects associated with aerodynamic forces, which will be discussed in Chapter 14.

At the Active Figures link at http://www.pse6.com, you can change launch angle and initial speed. You can also observe the changing components of velocity along the trajectory of the projectile.



Active Figure 4.7 The parabolic path of a projectile that leaves the origin with a velocity $\mathbf{v}_i$. The velocity vector $\mathbf{v}$ changes with time in both magnitude and direction. This change is the result of acceleration in the negative y direction. The x component of velocity remains constant in time because there is no acceleration along the horizontal direction. The y component of velocity is zero at the peak of the path.

PITFALL PREVENTION

4.2 Acceleration at the Highest Point

As discussed in Pitfall Prevention 2.8, many people claim that the acceleration of a projectile at the topmost point of its trajectory is zero. This mistake arises from confusion between zero vertical velocity and zero acceleration. If the projectile were to experience zero acceleration at the highest point, then its velocity at that point would not change—the projectile would move horizontally at constant speed from then on! This does not happen, because the acceleration is NOT zero anywhere along the trajectory.

The Telegraph Colour Library/Getty Images

A welder cuts holes through a heavy metal construction beam with a hot torch. The sparks generated in the process follow parabolic paths.

Repeating with the y component and using $y_i = 0$ and $a_y = -g$, we obtain

$$y_f = v_{yi}t + \tfrac{1}{2}a_y t^2 = (v_i \sin\theta_i)\,t - \tfrac{1}{2}gt^2 \tag{4.12}$$

Next, from Equation 4.11 we find $t = x_f/(v_i \cos\theta_i)$ and substitute this expression for t into Equation 4.12; this gives

$$y = (\tan\theta_i)\,x - \left(\frac{g}{2{v_i}^2\cos^2\theta_i}\right)x^2$$

This equation is valid for launch angles in the range $0 < \theta_i < \pi/2$. We have left the subscripts off the x and y because the equation is valid for any point (x, y) along the path of the projectile. The equation is of the form $y = ax - bx^2$, which is the equation of a parabola that passes through the origin. Thus, we have shown that the trajectory of a projectile is a parabola. Note that the trajectory is completely specified if both the initial speed v_i and the launch angle θ_i are known.

The vector expression for the position vector of the projectile as a function of time follows directly from Equation 4.9, with $\mathbf{a} = \mathbf{g}$:

$$\mathbf{r}_f = \mathbf{r}_i + \mathbf{v}_i t + \tfrac{1}{2}\mathbf{g}t^2$$

This expression is plotted in Figure 4.8, for a projectile launched from the origin, so that $\mathbf{r}_i = 0$.

The final position of a particle can be considered to be the superposition of the initial position $\mathbf{r}_i$, the term $\mathbf{v}_i t$, which is the displacement if no acceleration were present, and the term $\frac{1}{2}\mathbf{g}t^2$ that arises from the acceleration due to gravity. In other words, if there were no gravitational acceleration, the particle would continue to move along a straight path in the direction of $\mathbf{v}_i$. Therefore, the vertical distance $\frac{1}{2}\mathbf{g}t^2$ through which the particle "falls" off the straight-line path is the same distance that a freely falling object would fall during the same time interval.

In Section 4.2, we stated that two-dimensional motion with constant acceleration can be analyzed as a combination of two independent motions in the x and y directions, with accelerations a_x and a_y. Projectile motion is a special case of two-dimensional motion with constant acceleration and can be handled in this way, with zero acceleration in the x direction and $a_y = -g$ in the y direction. Thus, **when analyzing projectile motion, consider it to be the superposition of two motions:**

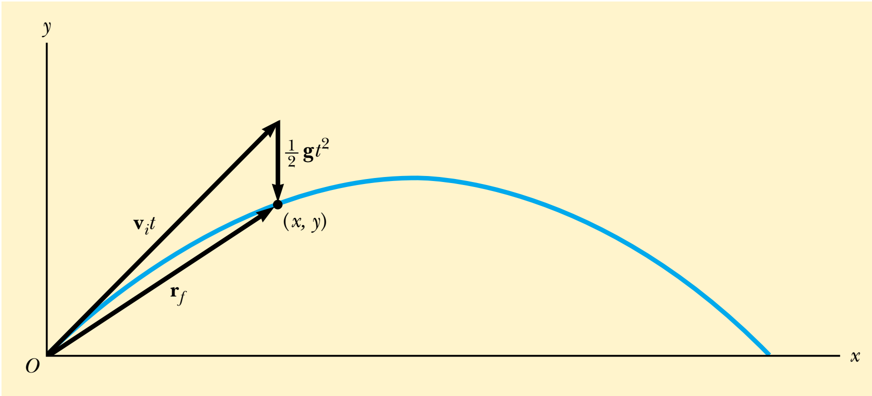


Figure 4.8 The position vector $\mathbf{r}_f$ of a projectile launched from the origin whose initial velocity at the origin is $\mathbf{v}_i$. The vector $\mathbf{v}_it$ would be the displacement of the projectile if gravity were absent, and the vector $\frac{1}{2}\mathbf{g}t^2$ is its vertical displacement due to its downward gravitational acceleration.

(1) **constant-velocity motion in the horizontal direction** and (2) **free-fall motion in the vertical direction.** The horizontal and vertical components of a projectile's motion are completely independent of each other and can be handled separately, with time t as the common variable for both components.

Quick Quiz 4.3 Suppose you are running at constant velocity and you wish to throw a ball such that you will catch it as it comes back down. In what direction should you throw the ball relative to you? (a) straight up (b) at an angle to the ground that depends on your running speed (c) in the forward direction.

Quick Quiz 4.4 As a projectile thrown upward moves in its parabolic path (such as in Figure 4.8), at what point along its path are the velocity and acceleration vectors for the projectile perpendicular to each other? (a) nowhere (b) the highest point (c) the launch point.

Quick Quiz 4.5 As the projectile in Quick Quiz 4.4 moves along its path, at what point are the velocity and acceleration vectors for the projectile parallel to each other? (a) nowhere (b) the highest point (c) the launch point.

Example 4.2 Approximating Projectile Motion

A ball is thrown in such a way that its initial vertical and horizontal components of velocity are 40 m/s and 20 m/s, respectively. Estimate the total time of flight and the distance the ball is from its starting point when it lands.

Solution A motion diagram like Figure 4.9 helps us *conceptualize* the problem. The phrase "A ball is thrown" allows us to *categorize* this as a projectile motion problem, which we *analyze* by continuing to study Figure 4.9. The acceleration vectors are all the same, pointing downward with a magnitude of nearly 10 m/s^2. The velocity vectors change direction. Their horizontal components are all the same: 20 m/s.

Remember that the two velocity components are independent of each other. By considering the vertical motion

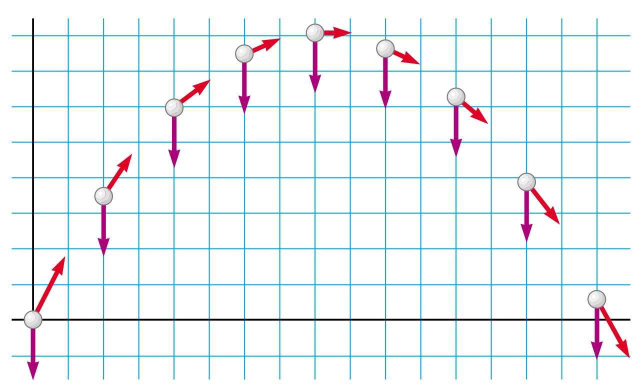


Figure 4.9 (Example 4.2) Motion diagram for a projectile.

first, we can determine how long the ball remains in the air. Because the vertical motion is free-fall, the vertical components of the velocity vectors change, second by second, from 40 m/s to roughly 30, 20, and 10 m/s in the upward direction, and then to 0 m/s. Subsequently, its velocity becomes 10, 20, 30, and 40 m/s in the downward direction. Thus it takes the ball about 4 s to go up and another 4 s to come back down, for a total time of flight of approximately 8 s.

Now we shift our analysis to the horizontal motion. Because the horizontal component of velocity is 20 m/s, and because the ball travels at this speed for 8 s, it ends up approximately 160 m from its starting point.

This is the first example that we have performed for projectile motion. In subsequent projectile motion problems, keep in mind the importance of separating the two components and of making approximations to give you rough expected results.

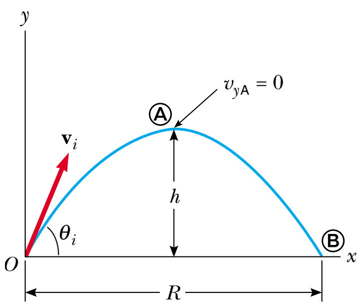


Figure 4.10 A projectile launched from the origin at $t_i = 0$ with an initial velocity $\mathbf{v}_i$. The maximum height of the projectile is h, and the horizontal range is R. At Ⓐ, the peak of the trajectory, the particle has coordinates $(R/2, h)$.

Horizontal Range and Maximum Height of a Projectile

Let us assume that a projectile is launched from the origin at $t_i = 0$ with a positive v_{yi} component, as shown in Figure 4.10. Two points are especially interesting to analyze: the peak point Ⓐ, which has Cartesian coordinates $(R/2, h)$, and the point Ⓑ, which has coordinates $(R, 0)$. The distance R is called the *horizontal range* of the projectile, and the distance h is its *maximum height*. Let us find h and R in terms of v_i, θ_i, and g.

We can determine h by noting that at the peak, $v_{yA} = 0$. Therefore, we can use Equation 4.8a to determine the time t_A at which the projectile reaches the peak:

$$v_{yf} = v_{yi} + a_y t$$

$$0 = v_i \sin\theta_i - g t_A$$

$$t_A = \frac{v_i \sin\theta_i}{g}$$

Substituting this expression for t_A into the y part of Equation 4.9a and replacing $y = y_A$ with h, we obtain an expression for h in terms of the magnitude and direction of the initial velocity vector:

$$h = (v_i \sin\theta_i)\frac{v_i \sin\theta_i}{g} - \tfrac{1}{2}g\left(\frac{v_i \sin\theta_i}{g}\right)^2$$

$$h = \frac{v_i^2 \sin^2\theta_i}{2g} \tag{4.13}$$

The range R is the horizontal position of the projectile at a time that is twice the time at which it reaches its peak, that is, at time $t_B = 2t_A$. Using the x part of Equation 4.9a, noting that $v_{xi} = v_{xB} = v_i \cos\theta_i$ and setting $x_B = R$ at $t = 2t_A$, we find that

$$R = v_{xi} t_B = (v_i \cos\theta_i) 2t_A$$

$$= (v_i \cos\theta_i)\frac{2v_i \sin\theta_i}{g} = \frac{2v_i^2 \sin\theta_i \cos\theta_i}{g}$$

Using the identity $\sin 2\theta = 2\sin\theta\cos\theta$ (see Appendix B.4), we write R in the more compact form

$$R = \frac{v_i^2 \sin 2\theta_i}{g} \tag{4.14}$$

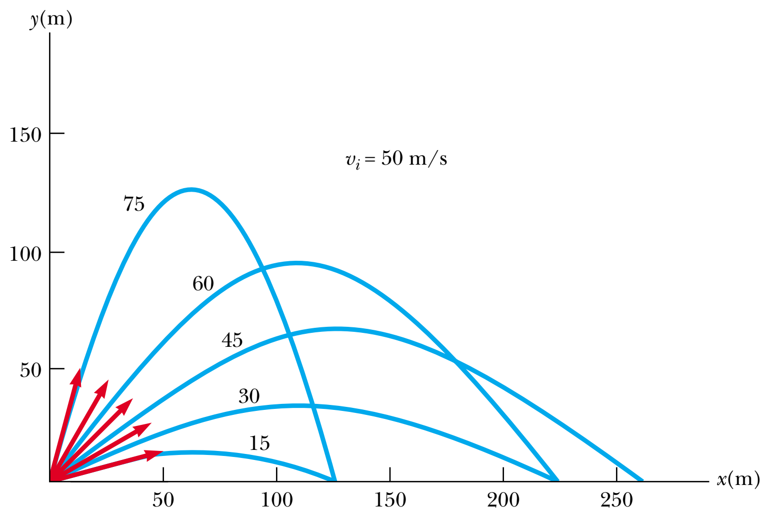
The maximum value of R from Equation 4.14 is $R_{max} = v_i^2/g$. This result follows from the fact that the maximum value of $\sin 2\theta_i$ is 1, which occurs when $2\theta_i = 90°$. Therefore, R is a maximum when $\theta_i = 45°$.

Figure 4.11 illustrates various trajectories for a projectile having a given initial speed but launched at different angles. As you can see, the range is a maximum for $\theta_i = 45°$. In addition, for any θ_i other than 45°, a point having Cartesian coordinates $(R, 0)$ can be reached by using either one of two complementary values of θ_i, such as 75° and 15°. Of course, the maximum height and time of flight for one of these values of θ_i are different from the maximum height and time of flight for the complementary value.

PITFALL PREVENTION

4.3 The Height and Range Equations

Equation 4.14 is useful for calculating R only for a symmetric path, as shown in Figure 4.10. If the path is not symmetric, *do not use this equation*. The general expressions given by Equations 4.8 and 4.9 are the *more important* results, because they give the position and velocity components of *any* particle moving in two dimensions at *any* time t.



Active Figure 4.11 A projectile launched from the origin with an initial speed of 50 m/s at various angles of projection. Note that complementary values of θ_i result in the same value of R (range of the projectile).

At the Active Figures link at http://www.pse6.com, you can vary the projection angle to observe the effect on the trajectory and measure the flight time.

Quick Quiz 4.6 Rank the launch angles for the five paths in Figure 4.11 with respect to time of flight, from the shortest time of flight to the longest.

PROBLEM-SOLVING HINTS

Projectile Motion

We suggest that you use the following approach to solving projectile motion problems:

- Select a coordinate system and resolve the initial velocity vector into x and y components.
- Follow the techniques for solving constant-velocity problems to analyze the horizontal motion. Follow the techniques for solving constant-acceleration problems to analyze the vertical motion. The x and y motions share the same time t.

Example 4.3 The Long Jump

A long-jumper (Fig. 4.12) leaves the ground at an angle of 20.0° above the horizontal and at a speed of 11.0 m/s.

(A) How far does he jump in the horizontal direction? (Assume his motion is equivalent to that of a particle.)

Solution We *conceptualize* the motion of the long-jumper as equivalent to that of a simple projectile such as the ball in Example 4.2, and *categorize* this problem as a projectile motion problem. Because the initial speed and launch angle are given, and because the final height is the same as the initial height, we further categorize this problem as satisfying the conditions for which Equations 4.13 and 4.14 can be used. This is the most direct way to *analyze* this problem, although the general methods that we have been describing will always give the correct answer. We will take the general approach and use components. Figure 4.10 provides a graphical representation of the flight of the long-jumper. As before, we set our origin of coordinates at the takeoff point and label the peak as Ⓐ and the landing point as Ⓑ. The horizontal motion is described by Equation 4.11:

$$x_f = x_{\text{B}} = (v_i \cos \theta_i)\, t_{\text{B}} = (11.0 \text{ m/s})(\cos 20.0°)\, t_{\text{B}}$$

The value of x_{B} can be found if the time of landing t_{B} is known. We can find t_{B} by remembering that $a_y = -g$ and by using the y part of Equation 4.8a. We also note that at the top of the jump the vertical component of velocity $v_{y\text{A}}$ is zero:

$$v_{yf} = v_{y\text{A}} = v_i \sin \theta_i - g t_{\text{A}}$$

$$0 = (11.0 \text{ m/s}) \sin 20.0° - (9.80 \text{ m/s}^2)\, t_{\text{A}}$$

$$t_{\text{A}} = 0.384 \text{ s}$$

Figure 4.12 (Example 4.3) Mike Powell, current holder of the world long jump record of 8.95 m.

This is the time at which the long-jumper is at the *top* of the jump. Because of the symmetry of the vertical motion, another 0.384 s passes before the jumper returns to the ground. Therefore, the time at which the jumper lands is $t_B = 2t_A = 0.768$ s. Substituting this value into the above expression for x_f gives

$$x_f = x_B = (11.0 \text{ m/s})(\cos 20.0°)(0.768 \text{ s}) = \boxed{7.94 \text{ m}}$$

This is a reasonable distance for a world-class athlete.

(B) What is the maximum height reached?

Solution We find the maximum height reached by using Equation 4.12:

$$\begin{aligned} y_{max} = y_A &= (v_i \sin \theta_i)t_A - \tfrac{1}{2}gt_A^2 \\ &= (11.0 \text{ m/s})(\sin 20.0°)(0.384 \text{ s}) \\ &\quad -\tfrac{1}{2}(9.80 \text{ m/s}^2)(0.384 \text{ s})^2 = \boxed{0.722 \text{ m}} \end{aligned}$$

To *finalize* this problem, find the answers to parts (A) and (B) using Equations 4.13 and 4.14. The results should agree. Treating the long-jumper as a particle is an oversimplification. Nevertheless, the values obtained are consistent with experience in sports. We learn that we can model a complicated system such as a long-jumper as a particle and still obtain results that are reasonable.

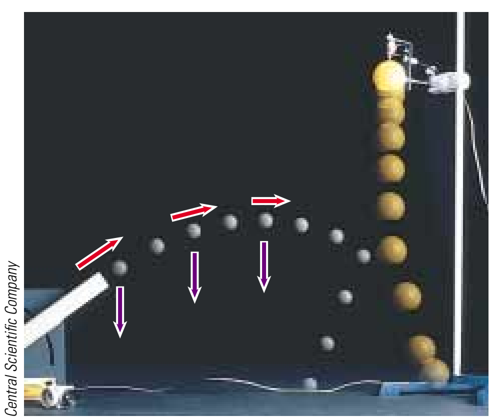
Example 4.4 A Bull's-Eye Every Time

Interactive

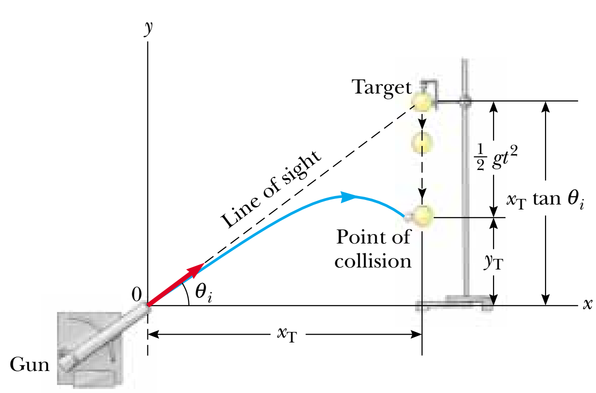
In a popular lecture demonstration, a projectile is fired at a target T in such a way that the projectile leaves the gun at the same time the target is dropped from rest, as shown in Figure 4.13. Show that if the gun is initially aimed at the stationary target, the projectile hits the target.

Solution *Conceptualize* the problem by studying Figure 4.13. Notice that the problem asks for no numbers. The expected result must involve an algebraic argument. Because both objects are subject only to gravity, we *categorize* this problem as one involving two objects in free-fall, one moving in one dimension and one moving in two. Let us now *analyze* the problem. A collision results under the conditions stated by noting that, as soon as they are released, the projectile and the target experience the same acceleration, $a_y = -g$. Figure 4.13b shows that the initial y coordinate of the target is $x_T \tan \theta_i$ and that it falls to a position $\frac{1}{2}gt^2$ below this coordinate at time t. Therefore, the y coordinate of the target at any moment after release is

$$y_T = x_T \tan \theta_i - \tfrac{1}{2}gt^2$$



(a)



(b)

Figure 4.13 (Example 4.4) (a) Multiflash photograph of projectile–target demonstration. If the gun is aimed directly at the target and is fired at the same instant the target begins to fall, the projectile will hit the target. Note that the velocity of the projectile (red arrows) changes in direction and magnitude, while its downward acceleration (violet arrows) remains constant. (b) Schematic diagram of the projectile–target demonstration. Both projectile and target have fallen through the same vertical distance at time t, because both experience the same acceleration $a_y = -g$.

Now if we use Equation 4.9a to write an expression for the y coordinate of the projectile at any moment, we obtain

$$y_P = x_P \tan \theta_i - \tfrac{1}{2}gt^2$$

Thus, by comparing the two previous equations, we see that when the y coordinates of the projectile and target are the same, their x coordinates are the same and a collision results. That is, when $y_P = y_T$, $x_P = x_T$. You can obtain the same result, using expressions for the position vectors for the projectile and target.

To *finalize* this problem, note that a collision can result only when $v_i \sin \theta_i \geq \sqrt{gd/2}$ where d is the initial elevation of the target above the floor. If $v_i \sin \theta_i$ is less than this value, the projectile will strike the floor before reaching the target.

Investigate this situation at the Interactive Worked Example link at http://www.pse6.com.

Example 4.5 That's Quite an Arm! Interactive

A stone is thrown from the top of a building upward at an angle of 30.0° to the horizontal with an initial speed of 20.0 m/s, as shown in Figure 4.14. If the height of the building is 45.0 m,

(A) how long does it take the stone to reach the ground?

Solution We *conceptualize* the problem by studying Figure 4.14, in which we have indicated the various parameters. By now, it should be natural to *categorize* this as a projectile motion problem.

To *analyze* the problem, let us once again separate motion into two components. The initial x and y components of the stone's velocity are

$$v_{xi} = v_i \cos \theta_i = (20.0 \text{ m/s}) \cos 30.0° = 17.3 \text{ m/s}$$

$$v_{yi} = v_i \sin \theta_i = (20.0 \text{ m/s}) \sin 30.0° = 10.0 \text{ m/s}$$

To find t, we can use $y_f = y_i + v_{yi}t + \frac{1}{2}a_y t^2$ (Eq. 4.9a) with $y_i = 0$, $y_f = -45.0$ m, $a_y = -g$, and $v_{yi} = 10.0$ m/s (there is a negative sign on the numerical value of y_f because we have chosen the top of the building as the origin):

$$-45.0 \text{ m} = (10.0 \text{ m/s})\,t - \tfrac{1}{2}(9.80 \text{ m/s}^2)\,t^2$$

Solving the quadratic equation for t gives, for the positive root, $t = \boxed{4.22 \text{ s.}}$ To *finalize* this part, think: Does the negative root have any physical meaning?

(B) What is the speed of the stone just before it strikes the ground?

Solution We can use Equation 4.8a, $v_{yf} = v_{yi} + a_y t$, with $t = 4.22$ s to obtain the y component of the velocity just before the stone strikes the ground:

$$v_{yf} = 10.0 \text{ m/s} - (9.80 \text{ m/s}^2)(4.22 \text{ s}) = -31.4 \text{ m/s}$$

Because $v_{xf} = v_{xi} = 17.3$ m/s, the required speed is

$$v_f = \sqrt{{v_{xf}}^2 + {v_{yf}}^2} = \sqrt{(17.3)^2 + (-31.4)^2} \text{ m/s} = \boxed{35.9 \text{ m/s}}$$

To *finalize* this part, is it reasonable that the y component of the final velocity is negative? Is it reasonable that the final speed is larger than the initial speed of 20.0 m/s?

What If? **What if a horizontal wind is blowing in the same direction as the ball is thrown and it causes the ball to have a horizontal acceleration component $a_x = 0.500$ m/s². Which part of this example, (A) or (B), will have a different answer?**

Answer Recall that the motions in the x and y directions are independent. Thus, the horizontal wind cannot affect the vertical motion. The vertical motion determines the time of the projectile in the air, so the answer to (A) does not change. The wind will cause the horizontal velocity component to increase with time, so that the final speed will change in part (B).

We can find the new final horizontal velocity component by using Equation 4.8a:

$$v_{xf} = v_{xi} + a_x t = 17.3 \text{ m/s} + (0.500 \text{ m/s}^2)(4.22 \text{ s})$$
$$= 19.4 \text{ m/s}$$

and the new final speed:

$$v_f = \sqrt{{v_{xf}}^2 + {v_{yf}}^2} = \sqrt{(19.4)^2 + (-31.4)^2} \text{ m/s} = 36.9 \text{ m/s}$$

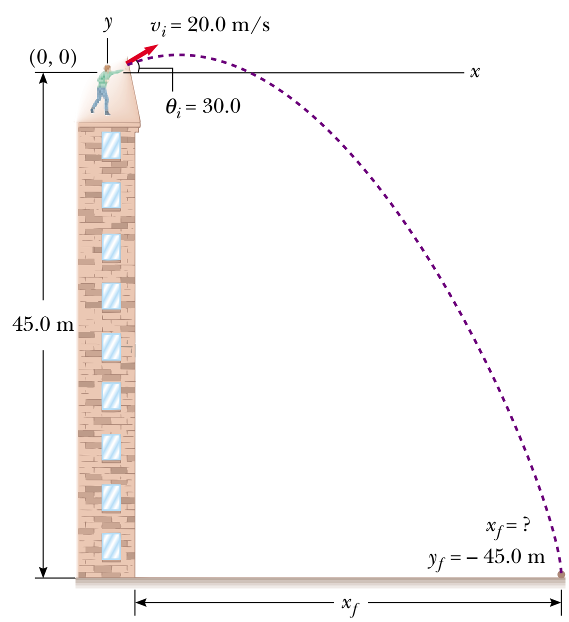


Figure 4.14 (Example 4.5) A stone is thrown from the top of a building.

Investigate this situation at the Interactive Worked Example link at http://www.pse6.com.

Example 4.6 The Stranded Explorers

A plane drops a package of supplies to a party of explorers, as shown in Figure 4.15. If the plane is traveling horizontally at 40.0 m/s and is 100 m above the ground, where does the package strike the ground relative to the point at which it is released?

Solution *Conceptualize* what is happening with the assistance of Figure 4.15. The plane is traveling horizontally when it drops the package. Because the package is in free-fall while moving in the horizontal direction, we *categorize* this as a projectile motion problem. To *analyze* the problem, we choose the coordinate system shown in Figure 4.15, in which the origin is at the point of release of the package. Consider first its horizontal motion. The only equation available for finding the position along the horizontal direction is $x_f = x_i + v_{xi}t$ (Eq. 4.9a). The initial x component of the package velocity is the same as that of the plane when the package is released: 40.0 m/s. Thus, we have

$$x_f = (40.0 \text{ m/s})t$$

If we know t, the time at which the package strikes the ground, then we can determine x_f, the distance the package travels in the horizontal direction. To find t, we use the equations that describe the vertical motion of the package. We know that, at the instant the package hits the ground, its y coordinate is $y_f = -100$ m. We also know that the initial vertical component of the package velocity v_{yi} is zero because at the moment of release, the package has only a horizontal component of velocity.

From Equation 4.9a, we have

$$y_f = -\tfrac{1}{2}gt^2$$
$$-100 \text{ m} = -\tfrac{1}{2}(9.80 \text{ m/s}^2)t^2$$
$$t = 4.52 \text{ s}$$

Substitution of this value for the time into the equation for the x coordinate gives

$$x_f = (40.0 \text{ m/s})(4.52 \text{ s}) = \boxed{181 \text{ m}}$$

The package hits the ground 181 m to the right of the drop point. To *finalize* this problem, we learn that an object dropped from a moving airplane does not fall straight down. It hits the ground at a point different from the one right below the plane where it was released. This was an important consideration for free-fall bombs such as those used in World War II.

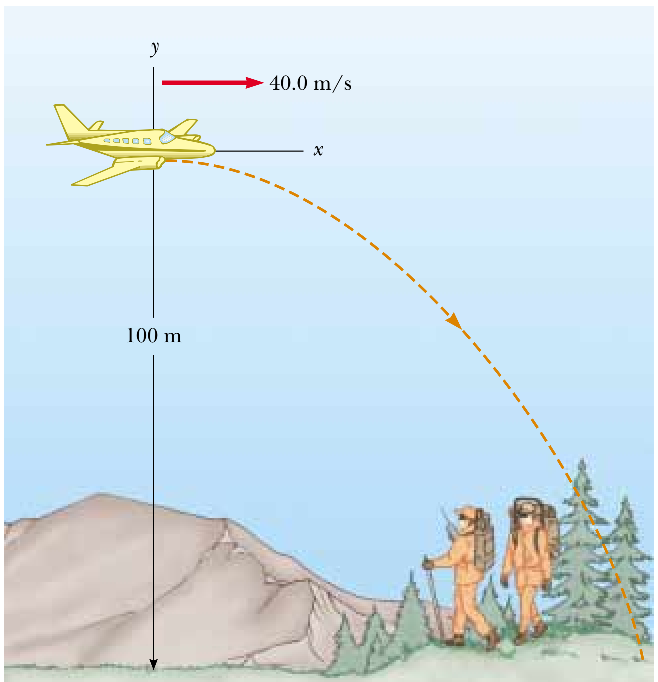


Figure 4.15 (Example 4.6) A package of emergency supplies is dropped from a plane to stranded explorers.

Example 4.7 The End of the Ski Jump

A ski-jumper leaves the ski track moving in the horizontal direction with a speed of 25.0 m/s, as shown in Figure 4.16. The landing incline below him falls off with a slope of 35.0°. Where does he land on the incline?

Solution We can *conceptualize* this problem based on observations of winter Olympic ski competitions. We observe the skier to be airborne for perhaps 4 s and go a distance of about 100 m horizontally. We should expect the value of d, the distance traveled along the incline, to be of the same order of magnitude. We *categorize* the problem as that of a particle in projectile motion.

To *analyze* the problem, it is convenient to select the beginning of the jump as the origin. Because $v_{xi} = 25.0$ m/s and $v_{yi} = 0$, the x and y component forms of Equation 4.9a are

$$(1) \qquad x_f = v_{xi}t = (25.0 \text{ m/s})t$$

$$(2) \qquad y_f = v_{yi}t + \tfrac{1}{2}a_y t^2 = -\tfrac{1}{2}(9.80 \text{ m/s}^2)t^2$$

From the right triangle in Figure 4.16, we see that the jumper's x and y coordinates at the landing point are $x_f = d\cos 35.0°$ and $y_f = -d\sin 35.0°$. Substituting these relationships into (1) and (2), we obtain

$$(3) \qquad d\cos 35.0° = (25.0 \text{ m/s})t$$

$$(4) \qquad -d\sin 35.0° = -\tfrac{1}{2}(9.80 \text{ m/s}^2)t^2$$

Solving (3) for t and substituting the result into (4), we find that $d = 109$ m. Hence, the x and y coordinates of the point at which the skier lands are

$$x_f = d\cos 35.0° = (109 \text{ m})\cos 35.0° = \boxed{89.3 \text{ m}}$$

$$y_f = -d\sin 35.0° = -(109 \text{ m})\sin 35.0° = \boxed{-62.5 \text{ m}}$$

To *finalize* the problem, let us compare these results to our expectations. We expected the horizontal distance to be on the order of 100 m, and our result of 89.3 m is indeed on

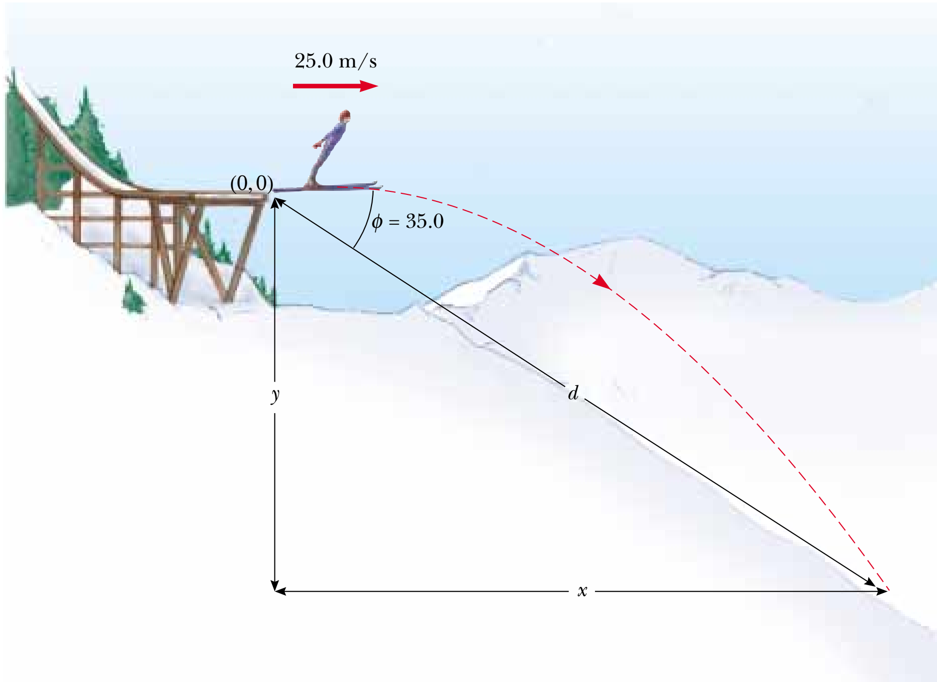


Figure 4.16 (Example 4.7) A ski jumper leaves the track moving in a horizontal direction.

this order of magnitude. It might be useful to calculate the time interval that the jumper is in the air and compare it to our estimate of about 4 s.

What If? **Suppose everything in this example is the same except that the ski jump is curved so that the jumper is projected upward at an angle from the end of the track. Is this a better design in terms of maximizing the length of the jump?**

Answer If the initial velocity has an upward component, the skier will be in the air longer, and should therefore travel further. However, tilting the initial velocity vector upward will reduce the horizontal component of the initial velocity. Thus, angling the end of the ski track upward at a *large* angle may actually *reduce* the distance. Consider the extreme case. The skier is projected at 90° to the horizontal, and simply goes up and comes back down at the end of the ski track! This argument suggests that there must be an optimal angle between 0 and 90° that represents a balance between making the flight time longer and the horizontal velocity component smaller.

We can find this optimal angle mathematically. We modify equations (1) through (4) in the following way, assuming that the skier is projected at an angle θ with respect to the horizontal:

$$(1) \text{ and } (3) \quad \rightarrow \quad x_f = (v_i \cos\theta)t = d\cos\phi$$

$$(2) \text{ and } (4) \quad \rightarrow \quad y_f = (v_i \sin\theta)t - \tfrac{1}{2}gt^2 = -d\sin\phi$$

If we eliminate the time t between these equations and then use differentiation to maximize d in terms of θ, we arrive (after several steps—see Problem 72!) at the following equation for the angle θ that gives the maximum value of d:

$$\theta = 45° - \frac{\phi}{2}$$

For the slope angle in Figure 4.16, $\phi = 35.0°$; this equation results in an optimal launch angle of $\theta = 27.5°$. Notice that for a slope angle of $\phi = 0°$, which represents a horizontal plane, this equation gives an optimal launch angle of $\theta = 45°$, as we would expect (see Figure 4.11).

4.4 Uniform Circular Motion

Figure 4.17a shows a car moving in a circular path with *constant speed v*. Such motion is called **uniform circular motion,** and occurs in many situations. It is often surprising to students to find that **even though an object moves at a constant speed in a circular path, it still has an acceleration.** To see why, consider the defining equation for average acceleration, $\overline{\mathbf{a}} = \Delta\mathbf{v}/\Delta t$ (Eq. 4.4).

Note that the acceleration depends on *the change in the velocity vector*. Because velocity is a vector quantity, there are two ways in which an acceleration can occur, as mentioned in Section 4.1: by a change in the *magnitude* of the velocity and/or by a change in the *direction* of the velocity. The latter situation occurs for an object moving with constant speed in a circular path. The velocity vector is always tangent to the

PITFALL PREVENTION

4.4 Acceleration of a Particle in Uniform Circular Motion

Remember that acceleration in physics is defined as a change in the *velocity,* not a change in the *speed* (contrary to the everyday interpretation). In circular motion, the velocity vector is changing in direction, so there is indeed an acceleration.

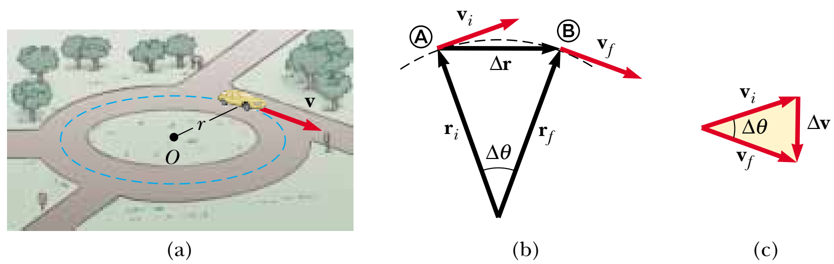


Figure 4.17 (a) A car moving along a circular path at constant speed experiences uniform circular motion. (b) As a particle moves from Ⓐ to Ⓑ, its velocity vector changes from $\mathbf{v}_i$ to $\mathbf{v}_f$. (c) The construction for determining the direction of the change in velocity $\Delta\mathbf{v}$, which is toward the center of the circle for small $\Delta\mathbf{r}$.

path of the object and perpendicular to the radius of the circular path. We now show that the acceleration vector in uniform circular motion is always perpendicular to the path and always points toward the center of the circle. An acceleration of this nature is called a **centripetal acceleration** (*centripetal* means *center-seeking*), and its magnitude is

Centripetal acceleration

$$a_c = \frac{v^2}{r} \tag{4.15}$$

where r is the radius of the circle. The subscript on the acceleration symbol reminds us that the acceleration is centripetal.

First note that the acceleration must be perpendicular to the path followed by the object, which we will model as a particle. If this were not true, there would be a component of the acceleration parallel to the path and, therefore, parallel to the velocity vector. Such an acceleration component would lead to a change in the speed of the particle along the path. But this is inconsistent with our setup of the situation—the particle moves with constant speed along the path. Thus, for *uniform* circular motion, the acceleration vector can only have a component perpendicular to the path, which is toward the center of the circle.

To derive Equation 4.15, consider the diagram of the position and velocity vectors in Figure 4.17b. In addition, the figure shows the vector representing the change in position $\Delta\mathbf{r}$. The particle follows a circular path, part of which is shown by the dotted curve. The particle is at Ⓐ at time t_i, and its velocity at that time is $\mathbf{v}_i$; it is at Ⓑ at some later time t_f, and its velocity at that time is $\mathbf{v}_f$. Let us also assume that $\mathbf{v}_i$ and $\mathbf{v}_f$ differ only in direction; their magnitudes are the same (that is, $v_i = v_f = v$, because it is *uniform* circular motion). In order to calculate the acceleration of the particle, let us begin with the defining equation for average acceleration (Eq. 4.4):

$$\overline{\mathbf{a}} \equiv \frac{\mathbf{v}_f - \mathbf{v}_i}{t_f - t_i} = \frac{\Delta\mathbf{v}}{\Delta t}$$

In Figure 4.17c, the velocity vectors in Figure 4.17b have been redrawn tail to tail. The vector $\Delta\mathbf{v}$ connects the tips of the vectors, representing the vector addition $\mathbf{v}_f = \mathbf{v}_i + \Delta\mathbf{v}$. In both Figures 4.17b and 4.17c, we can identify triangles that help us analyze the motion. The angle $\Delta\theta$ between the two position vectors in Figure 4.17b is the same as the angle between the velocity vectors in Figure 4.17c, because the velocity vector $\mathbf{v}$ is always perpendicular to the position vector $\mathbf{r}$. Thus, the two triangles are *similar*. (Two triangles are similar if the angle between any two sides is the same for both triangles and if the ratio of the lengths of these sides is the same.) This enables us to write a relationship between the lengths of the sides for the two triangles:

$$\frac{|\Delta\mathbf{v}|}{v} = \frac{|\Delta\mathbf{r}|}{r}$$

where $v = v_i = v_f$ and $r = r_i = r_f$. This equation can be solved for $|\Delta\mathbf{v}|$ and the expression so obtained can be substituted into $\overline{\mathbf{a}} = \Delta\mathbf{v}/\Delta t$ to give the magnitude of the average acceleration over the time interval for the particle to move from Ⓐ to Ⓑ:

$$|\overline{\mathbf{a}}| = \frac{|\Delta\mathbf{v}|}{\Delta t} = \frac{v}{r}\frac{|\Delta\mathbf{r}|}{\Delta t}$$

Now imagine that points Ⓐ and Ⓑ in Figure 4.17b become extremely close together. As Ⓐ and Ⓑ approach each other, Δt approaches zero, and the ratio $|\Delta\mathbf{r}|/\Delta t$ approaches the speed v. In addition, the average acceleration becomes the instantaneous acceleration at point Ⓐ. Hence, in the limit $\Delta t \rightarrow 0$, the magnitude of the acceleration is

$$a_c = \frac{v^2}{r}$$

Thus, in uniform circular motion the acceleration is directed inward toward the center of the circle and has magnitude v^2/r.

In many situations it is convenient to describe the motion of a particle moving with constant speed in a circle of radius r in terms of the **period** T, which is defined as the time required for one complete revolution. In the time interval T the particle moves a distance of $2\pi r$, which is equal to the circumference of the particle's circular path. Therefore, because its speed is equal to the circumference of the circular path divided by the period, or $v = 2\pi r/T$, it follows that

Period of circular motion

$$T \equiv \frac{2\pi r}{v} \tag{4.16}$$

PITFALL PREVENTION

4.5 Centripetal Acceleration is not Constant

We derived the magnitude of the centripetal acceleration vector and found it to be constant for uniform circular motion. But *the centripetal acceleration vector is not constant.* It always points toward the center of the circle, but continuously changes direction as the object moves around the circular path.

Quick Quiz 4.7 Which of the following correctly describes the centripetal acceleration vector for a particle moving in a circular path? (a) constant and always perpendicular to the velocity vector for the particle (b) constant and always parallel to the velocity vector for the particle (c) of constant magnitude and always perpendicular to the velocity vector for the particle (d) of constant magnitude and always parallel to the velocity vector for the particle.

Quick Quiz 4.8 A particle moves in a circular path of radius r with speed v. It then increases its speed to $2v$ while traveling along the same circular path. The centripetal acceleration of the particle has changed by a factor of (a) 0.25 (b) 0.5 (c) 2 (d) 4 (e) impossible to determine

Example 4.8 The Centripetal Acceleration of the Earth

What is the centripetal acceleration of the Earth as it moves in its orbit around the Sun?

Solution We *conceptualize* this problem by bringing forth our familiar mental image of the Earth in a circular orbit around the Sun. We will simplify the problem by modeling the Earth as a particle and approximating the Earth's orbit as circular (it's actually slightly elliptical). This allows us to *categorize* this problem as that of a particle in uniform circular motion. When we begin to *analyze* this problem, we realize that we do not know the orbital speed of the Earth in Equation 4.15. With the help of Equation 4.16, however, we can recast Equation 4.15 in terms of the period of the Earth's orbit, which we know is one year:

$$a_c = \frac{v^2}{r} = \frac{\left(\dfrac{2\pi r}{T}\right)^2}{r} = \frac{4\pi^2 r}{T^2}$$

$$= \frac{4\pi^2(1.496 \times 10^{11}\ \text{m})}{(1\ \text{yr})^2}\left(\frac{1\ \text{yr}}{3.156 \times 10^7\ \text{s}}\right)^2$$

$$= \boxed{5.93 \times 10^{-3}\ \text{m/s}^2}$$

To *finalize* this problem, note that this acceleration is much smaller than the free-fall acceleration on the surface of the Earth. An important thing we learned here is the technique of replacing the speed v in terms of the period T of the motion.

4.5 Tangential and Radial Acceleration

Let us consider the motion of a particle along a smooth curved path where the velocity changes both in direction and in magnitude, as described in Figure 4.18. In this situation, the velocity vector is always tangent to the path; however, the acceleration vector **a** is at some angle to the path. At each of three points Ⓐ, Ⓑ, and Ⓒ in Figure 4.18, we draw dashed circles that represent a portion of the actual path at each point. The radius of the circles is equal to the radius of curvature of the path at each point.

As the particle moves along the curved path in Figure 4.18, the direction of the total acceleration vector **a** changes from point to point. This vector can be resolved into two components, based on an origin at the center of the dashed circle: a radial component a_r along the radius of the model circle, and a tangential component a_t perpendicular to this radius. The *total* acceleration vector **a** can be written as the vector sum of the component vectors:

Total acceleration

$$\mathbf{a} = \mathbf{a}_r + \mathbf{a}_t \tag{4.17}$$

The tangential acceleration component causes the change in the speed of the particle. This component is parallel to the instantaneous velocity, and is given by

Tangential acceleration

$$a_t = \frac{d|\mathbf{v}|}{dt} \tag{4.18}$$

The radial acceleration component arises from the change in direction of the velocity vector and is given by

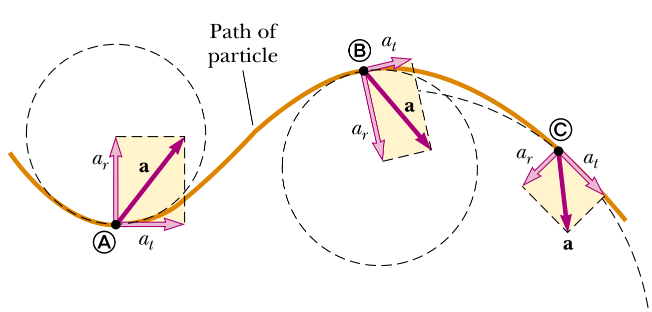
Radial acceleration

$$a_r = -a_c = -\frac{v^2}{r} \tag{4.19}$$

where r is the radius of curvature of the path at the point in question. We recognize the radial component of the acceleration as the centripetal acceleration discussed in Section 4.4. The negative sign indicates that the direction of the centripetal acceleration is toward the center of the circle representing the radius of curvature, which is opposite the direction of the radial unit vector $\hat{\mathbf{r}}$, which always points away from the center of the circle.

Because $\mathbf{a}_r$ and $\mathbf{a}_t$ are perpendicular component vectors of **a**, it follows that the magnitude of **a** is $a = \sqrt{a_r^2 + a_t^2}$. At a given speed, a_r is large when the radius of curvature is small (as at points Ⓐ and Ⓑ in Fig. 4.18) and small when r is large (such as at point Ⓒ). The direction of $\mathbf{a}_t$ is either in the same direction as **v** (if v is increasing) or opposite **v** (if v is decreasing).

In uniform circular motion, where v is constant, $a_t = 0$ and the acceleration is always completely radial, as we described in Section 4.4. In other words, uniform circular motion is a special case of motion along a general curved path. Furthermore, if the direction of **v** does not change, then there is no radial acceleration and the motion is one-dimensional (in this case, $a_r = 0$, but a_t may not be zero).



Active Figure 4.18 The motion of a particle along an arbitrary curved path lying in the xy plane. If the velocity vector **v** (always tangent to the path) changes in direction and magnitude, the components of the acceleration **a** are a tangential component a_t and a radial component a_r.

***At the Active Figures link* at http://www.pse6.com, you can study the acceleration components of a roller coaster car.**

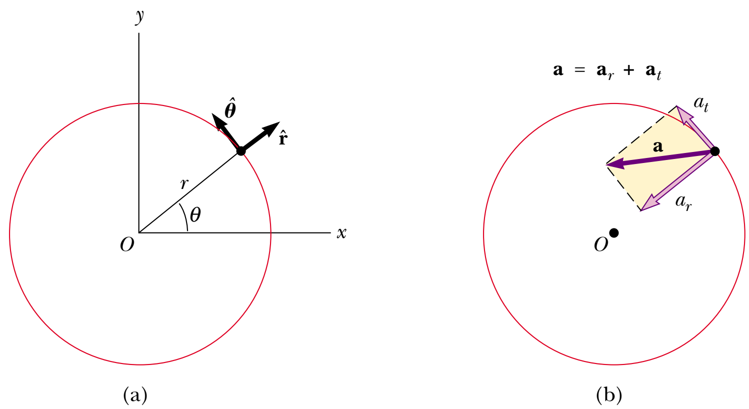


Figure 4.19 (a) Descriptions of the unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$. (b) The total acceleration **a** of a particle moving along a curved path (which at any instant is part of a circle of radius r) is the sum of radial and tangential component vectors. The radial component vector is directed toward the center of curvature. If the tangential component of acceleration becomes zero, the particle follows uniform circular motion.

It is convenient to write the acceleration of a particle moving in a circular path in terms of unit vectors. We do this by defining the unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ shown in Figure 4.19a, where $\hat{\mathbf{r}}$ is a unit vector lying along the radius vector and directed radially outward from the center of the circle and $\hat{\boldsymbol{\theta}}$ is a unit vector tangent to the circle. The direction of $\hat{\boldsymbol{\theta}}$ is in the direction of increasing θ, where θ is measured counterclockwise from the positive x axis. Note that both $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ "move along with the particle" and so vary in time. Using this notation, we can express the total acceleration as

$$\mathbf{a} = \mathbf{a}_t + \mathbf{a}_r = \frac{d|\mathbf{v}|}{dt}\hat{\boldsymbol{\theta}} - \frac{v^2}{r}\hat{\mathbf{r}} \tag{4.20}$$

These vectors are described in Figure 4.19b.

Quick Quiz 4.9 A particle moves along a path and its speed increases with time. In which of the following cases are its acceleration and velocity vectors parallel? (a) the path is circular (b) the path is straight (c) the path is a parabola (d) never.

Quick Quiz 4.10 A particle moves along a path and its speed increases with time. In which of the following cases are its acceleration and velocity vectors perpendicular everywhere along the path? (a) the path is circular (b) the path is straight (c) the path is a parabola (d) never.

Example 4.9 Over the Rise

A car exhibits a constant acceleration of 0.300 m/s^2 parallel to the roadway. The car passes over a rise in the roadway such that the top of the rise is shaped like a circle of radius 500 m. At the moment the car is at the top of the rise, its velocity vector is horizontal and has a magnitude of 6.00 m/s. What is the direction of the total acceleration vector for the car at this instant?

Solution *Conceptualize* the situation using Figure 4.20a. Because the car is moving along a curved path, we can *categorize* this as a problem involving a particle experiencing both tangential and radial acceleration. Now we recognize that this is a relatively simple plug-in problem. The radial acceleration is given by Equation 4.19. With $v = 6.00$ m/s and $r = 500$ m, we find that

$$a_r = -\frac{v^2}{r} = -\frac{(6.00\text{ m/s})^2}{500\text{ m}} = \boxed{-0.0720\text{ m/s}^2}$$

The radial acceleration vector is directed straight downward

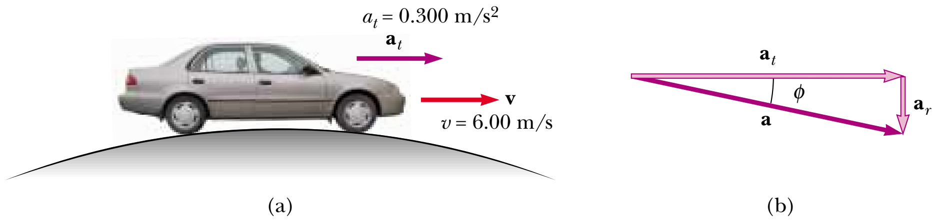


Figure 4.20 (Example 4.9) (a) A car passes over a rise that is shaped like a circle. (b) The total acceleration vector **a** is the sum of the tangential and radial acceleration vectors $\mathbf{a}_t$ and $\mathbf{a}_r$.

while the tangential acceleration vector has magnitude 0.300 m/s^2 and is horizontal. Because $\mathbf{a} = \mathbf{a}_r + \mathbf{a}_t$, the magnitude of **a** is

$$a = \sqrt{a_r{}^2 + a_t{}^2} = \sqrt{(-0.0720)^2 + (0.300)^2}\ \text{m/s}^2$$

$$= \boxed{0.309\ \text{m/s}^2}$$

If ϕ is the angle between **a** and the horizontal, then

$$\phi = \tan^{-1}\frac{a_r}{a_t} = \tan^{-1}\left(\frac{-0.0720\ \text{m/s}^2}{0.300\ \text{m/s}^2}\right) = \boxed{-13.5°}$$

This angle is measured downward from the horizontal. (See Figure 4.20b.)

4.6 Relative Velocity and Relative Acceleration

In this section, we describe how observations made by different observers in different frames of reference are related to each other. We find that observers in different frames of reference may measure different positions, velocities, and accelerations for a given particle. That is, two observers moving relative to each other generally do not agree on the outcome of a measurement.

As an example, consider two observers watching a man walking on a moving beltway at an airport in Figure 4.21. The woman standing on the moving beltway will see the man moving at a normal walking speed. The woman observing from the stationary floor will see the man moving with a higher speed, because the beltway speed combines with his walking speed. Both observers look at the same man and arrive at different values for his speed. Both are correct; the difference in their measurements is due to the relative velocity of their frames of reference.

Suppose a person riding on a skateboard (observer A) throws a ball in such a way that it appears in this person's frame of reference to move first straight upward and

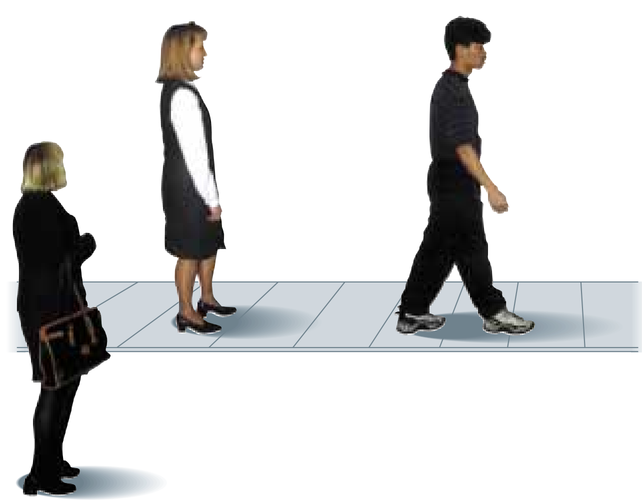


Figure 4.21 Two observers measure the speed of a man walking on a moving beltway. The woman standing on the beltway sees the man moving with a slower speed than the woman observing from the stationary floor.

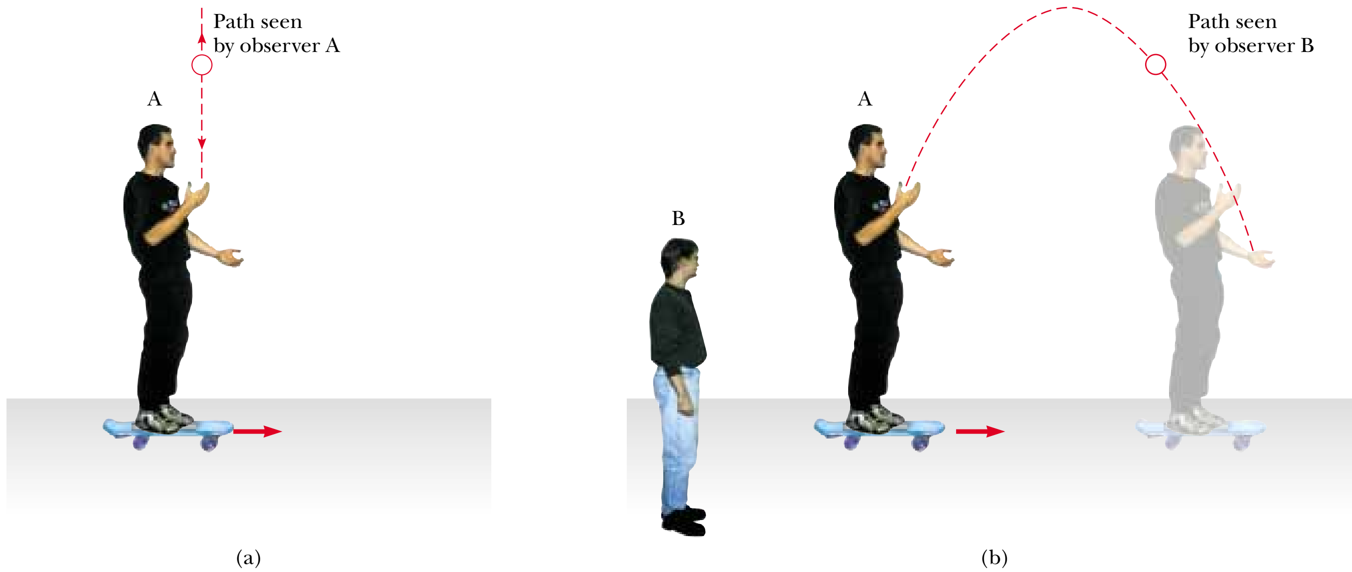


Figure 4.22 (a) Observer A on a moving skateboard throws a ball upward and sees it rise and fall in a straight-line path. (b) Stationary observer B sees a parabolic path for the same ball.

then straight downward along the same vertical line, as shown in Figure 4.22a. An observer B on the ground sees the path of the ball as a parabola, as illustrated in Figure 4.22b. Relative to observer B, the ball has a vertical component of velocity (resulting from the initial upward velocity and the downward acceleration due to gravity) *and* a horizontal component.

Another example of this concept is the motion of a package dropped from an airplane flying with a constant velocity—a situation we studied in Example 4.6. An observer on the airplane sees the motion of the package as a straight line downward toward Earth. The stranded explorer on the ground, however, sees the trajectory of the package as a parabola. Once the package is dropped, and the airplane continues to move horizontally with the same velocity, the package hits the ground directly beneath the airplane (if we assume that air resistance is neglected)!

In a more general situation, consider a particle located at point Ⓐ in Figure 4.23. Imagine that the motion of this particle is being described by two observers, one in reference frame S, fixed relative to Earth, and another in reference frame S', moving to the right relative to S (and therefore relative to Earth) with a constant velocity $\mathbf{v}_0$. (Relative to an observer in S', S moves to the left with a velocity $-\mathbf{v}_0$.) Where an observer stands in a reference frame is irrelevant in this discussion, but for purposes of this discussion let us place each observer at her or his respective origin.

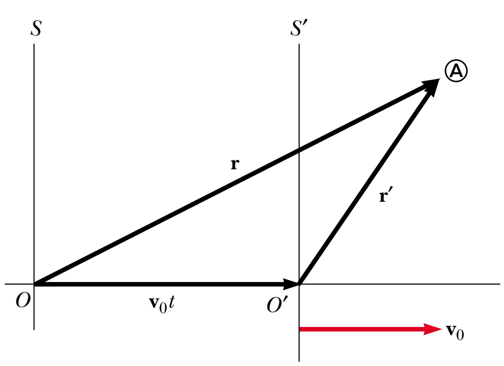


Figure 4.23 A particle located at Ⓐ is described by two observers, one in the fixed frame of reference S, and the other in the frame S', which moves to the right with a constant velocity $\mathbf{v}_0$. The vector $\mathbf{r}$ is the particle's position vector relative to S, and $\mathbf{r}'$ is its position vector relative to S'.

We define the time $t = 0$ as that instant at which the origins of the two reference frames coincide in space. Thus, at time t, the origins of the reference frames will be separated by a distance $v_0 t$. We label the position of the particle relative to the S frame with the position vector $\mathbf{r}$ and that relative to the S' frame with the position vector $\mathbf{r}'$, both at time t. The vectors $\mathbf{r}$ and $\mathbf{r}'$ are related to each other through the expression $\mathbf{r} = \mathbf{r}' + \mathbf{v}_0 t$, or

Galilean coordinate transformation

$$\mathbf{r}' = \mathbf{r} - \mathbf{v}_0 t \tag{4.21}$$

If we differentiate Equation 4.21 with respect to time and note that $\mathbf{v}_0$ is constant, we obtain

$$\frac{d\mathbf{r}'}{dt} = \frac{d\mathbf{r}}{dt} - \mathbf{v}_0$$

Galilean velocity transformation

$$\mathbf{v}' = \mathbf{v} - \mathbf{v}_0 \tag{4.22}$$

where $\mathbf{v}'$ is the velocity of the particle observed in the S' frame and $\mathbf{v}$ is its velocity observed in the S frame. Equations 4.21 and 4.22 are known as **Galilean transformation equations.** They relate the position and velocity of a particle as measured by observers in relative motion.

Although observers in two frames measure different velocities for the particle, they measure the *same acceleration* when $\mathbf{v}_0$ is constant. We can verify this by taking the time derivative of Equation 4.22:

$$\frac{d\mathbf{v}'}{dt} = \frac{d\mathbf{v}}{dt} - \frac{d\mathbf{v}_0}{dt}$$

Because $\mathbf{v}_0$ is constant, $d\mathbf{v}_0/dt = 0$. Therefore, we conclude that $\mathbf{a}' = \mathbf{a}$ because $\mathbf{a}' = d\mathbf{v}'/dt$ and $\mathbf{a} = d\mathbf{v}/dt$. That is, **the acceleration of the particle measured by an observer in one frame of reference is the same as that measured by any other observer moving with constant velocity relative to the first frame.**

Quick Quiz 4.11 A passenger, observer A, in a car traveling at a constant horizontal velocity of magnitude 60 mi/h pours a cup of coffee for the tired driver. Observer B stands on the side of the road and watches the pouring process through the window of the car as it passes. Which observer(s) sees a parabolic path for the coffee as it moves through the air? (a) A (b) B (c) both A and B (d) neither A nor B.

Example 4.10 A Boat Crossing a River

A boat heading due north crosses a wide river with a speed of 10.0 km/h relative to the water. The water in the river has a uniform speed of 5.00 km/h due east relative to the Earth. Determine the velocity of the boat relative to an observer standing on either bank.

Solution To *conceptualize* this problem, imagine moving across a river while the current pushes you along the river. You will not be able to move directly across the river, but will end up downstream, as suggested in Figure 4.24. Because of the separate velocities of you and the river, we can *categorize* this as a problem involving relative velocities. We will *analyze* this problem with the techniques discussed in this section. We know $\mathbf{v}_{br}$, the velocity of the *boat* relative to the *river*, and $\mathbf{v}_{rE}$, the velocity of the *river* relative to *Earth*. What we must find is $\mathbf{v}_{bE}$, the velocity of the *boat* relative to *Earth*. The relationship between these three quantities is

$$\mathbf{v}_{bE} = \mathbf{v}_{br} + \mathbf{v}_{rE}$$

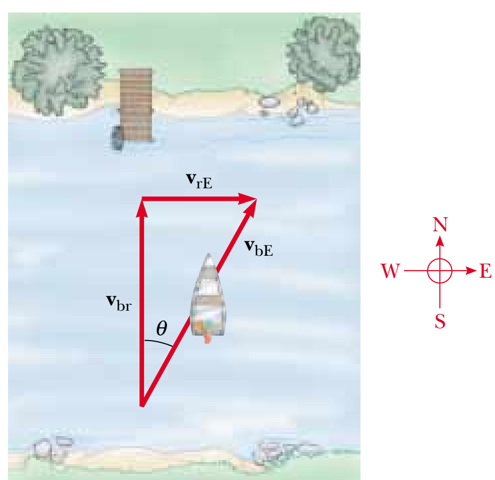


Figure 4.24 (Example 4.10) A boat aims directly across a river and ends up downstream.

The terms in the equation must be manipulated as vector quantities; the vectors are shown in Figure 4.24. The quantity $\mathbf{v}_{br}$ is due north, $\mathbf{v}_{rE}$ is due east, and the vector sum of the two, $\mathbf{v}_{bE}$, is at an angle θ, as defined in Figure 4.24. Thus, we can find the speed v_{bE} of the boat relative to Earth by using the Pythagorean theorem:

$$v_{bE} = \sqrt{v_{br}^2 + v_{rE}^2} = \sqrt{(10.0)^2 + (5.00)^2}\ \text{km/h}$$
$$= 11.2\ \text{km/h}$$

The direction of $\mathbf{v}_{bE}$ is

$$\theta = \tan^{-1}\left(\frac{v_{rE}}{v_{br}}\right) = \tan^{-1}\left(\frac{5.00}{10.0}\right) = 26.6°$$

The boat is moving at a speed of 11.2 km/h in the direction 26.6° east of north relative to Earth. To *finalize* the problem, note that the speed of 11.2 km/h is faster than your boat speed of 10.0 km/h. The current velocity adds to yours to give you a larger speed. Notice in Figure 4.24 that your resultant velocity is at an angle to the direction straight across the river, so you will end up downstream, as we predicted.

Example 4.11 Which Way Should We Head?

If the boat of the preceding example travels with the same speed of 10.0 km/h relative to the river and is to travel due north, as shown in Figure 4.25, what should its heading be?

Solution This example is an extension of the previous one, so we have already *conceptualized* and *categorized* the problem. The *analysis* now involves the new triangle shown in Figure 4.25. As in the previous example, we know $\mathbf{v}_{rE}$ and the magnitude of the vector $\mathbf{v}_{br}$, and we want $\mathbf{v}_{bE}$ to be directed across the river. Note the difference between the triangle in Figure 4.24 and the one in Figure 4.25—the hypotenuse in Figure 4.25 is no longer $\mathbf{v}_{bE}$. Therefore, when we use the Pythagorean theorem to find $\mathbf{v}_{bE}$ in this situation, we obtain

$$v_{bE} = \sqrt{v_{br}^2 - v_{rE}^2} = \sqrt{(10.0)^2 - (5.00)^2}\ \text{km/h} = 8.66\ \text{km/h}$$

Now that we know the magnitude of $\mathbf{v}_{bE}$, we can find the direction in which the boat is heading:

$$\theta = \tan^{-1}\left(\frac{v_{rE}}{v_{bE}}\right) = \tan^{-1}\left(\frac{5.00}{8.66}\right) = 30.0°$$

To *finalize* this problem, we learn that the boat must head upstream in order to travel directly northward across the river. For the given situation, the boat must steer a course 30.0° west of north.

What If? **Imagine that the two boats in Examples 4.10 and 4.11 are racing across the river. Which boat arrives at the opposite bank first?**

Answer In Example 4.10, the velocity of 10 km/h is aimed directly across the river. In Example 4.11, the velocity that is directed across the river has a magnitude of only 8.66 km/h. Thus, the boat in Example 4.10 has a larger velocity component directly across the river and will arrive first.

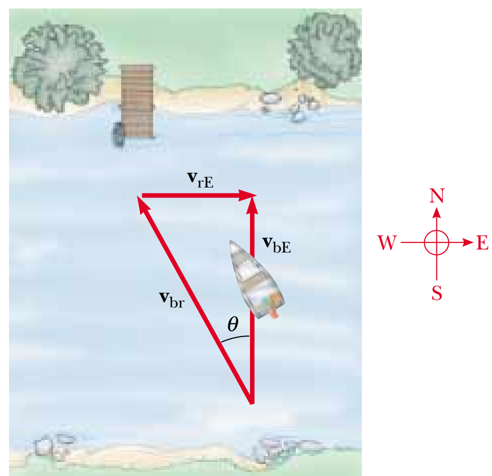


Figure 4.25 (Example 4.11) To move directly across the river, the boat must aim upstream.

SUMMARY

Take a practice test for this chapter by clicking on the Practice Test link at http://www.pse6.com.

If a particle moves with *constant* acceleration $\mathbf{a}$ and has velocity $\mathbf{v}_i$ and position $\mathbf{r}_i$ at $t = 0$, its velocity and position vectors at some later time t are

$$\mathbf{v}_f = \mathbf{v}_i + \mathbf{a}t \quad (4.8)$$

$$\mathbf{r}_f = \mathbf{r}_i + \mathbf{v}_i t + \tfrac{1}{2}\mathbf{a}t^2 \quad (4.9)$$

For two-dimensional motion in the *xy* plane under constant acceleration, each of these vector expressions is equivalent to two component expressions—one for the motion in the *x* direction and one for the motion in the *y* direction.

Projectile motion is one type of two-dimensional motion under constant acceleration, where $a_x = 0$ and $a_y = -g$. It is useful to think of projectile motion as the superposition of two motions: (1) constant-velocity motion in the x direction and

(2) free-fall motion in the vertical direction subject to a constant downward acceleration of magnitude $g = 9.80 \text{ m/s}^2$.

A particle moving in a circle of radius r with constant speed v is in **uniform circular motion.** It undergoes a radial acceleration $\mathbf{a}_r$ because the direction of $\mathbf{v}$ changes in time. The magnitude of $\mathbf{a}_r$ is the **centripetal acceleration** a_c:

$$a_c = \frac{v^2}{r} \tag{4.19}$$

and its direction is always toward the center of the circle.

If a particle moves along a curved path in such a way that both the magnitude and the direction of $\mathbf{v}$ change in time, then the particle has an acceleration vector that can be described by two component vectors: (1) a radial component vector $\mathbf{a}_r$ that causes the change in direction of $\mathbf{v}$ and (2) a tangential component vector $\mathbf{a}_t$ that causes the change in magnitude of $\mathbf{v}$. The magnitude of $\mathbf{a}_r$ is v^2/r, and the magnitude of $\mathbf{a}_t$ is $d|\mathbf{v}|/dt$.

The velocity $\mathbf{v}$ of a particle measured in a fixed frame of reference S can be related to the velocity $\mathbf{v}'$ of the same particle measured in a moving frame of reference S' by

$$\mathbf{v}' = \mathbf{v} - \mathbf{v}_0 \tag{4.22}$$

where $\mathbf{v}_0$ is the velocity of S' relative to S.

QUESTIONS

1. Can an object accelerate if its speed is constant? Can an object accelerate if its velocity is constant?

2. If you know the position vectors of a particle at two points along its path and also know the time it took to move from one point to the other, can you determine the particle's instantaneous velocity? Its average velocity? Explain.

3. Construct motion diagrams showing the velocity and acceleration of a projectile at several points along its path if (a) the projectile is launched horizontally and (b) the projectile is launched at an angle θ with the horizontal.

4. A baseball is thrown with an initial velocity of $(10\hat{\mathbf{i}} + 15\hat{\mathbf{j}})$ m/s. When it reaches the top of its trajectory, what are (a) its velocity and (b) its acceleration? Neglect the effect of air resistance.

5. A baseball is thrown such that its initial x and y components of velocity are known. Neglecting air resistance, describe how you would calculate, at the instant the ball reaches the top of its trajectory, (a) its position, (b) its velocity, and (c) its acceleration. How would these results change if air resistance were taken into account?

6. A spacecraft drifts through space at a constant velocity. Suddenly a gas leak in the side of the spacecraft gives it a constant acceleration in a direction perpendicular to the initial velocity. The orientation of the spacecraft does not change, so that the acceleration remains perpendicular to the original direction of the velocity. What is the shape of the path followed by the spacecraft in this situation?

7. A ball is projected horizontally from the top of a building. One second later another ball is projected horizontally from the same point with the same velocity. At what point in the motion will the balls be closest to each other? Will the first ball always be traveling faster than the second ball? What will be the time interval between when the balls hit the ground? Can the horizontal projection velocity of the second ball be changed so that the balls arrive at the ground at the same time?

8. A rock is dropped at the same instant that a ball, at the same initial elevation, is thrown horizontally. Which will have the greater speed when it reaches ground level?

9. Determine which of the following moving objects obey the equations of projectile motion developed in this chapter. (a) A ball is thrown in an arbitrary direction. (b) A jet airplane crosses the sky with its engines thrusting the plane forward. (c) A rocket leaves the launch pad. (d) A rocket moving through the sky after its engines have failed. (e) A stone is thrown under water.

10. How can you throw a projectile so that it has zero speed at the top of its trajectory? So that it has nonzero speed at the top of its trajectory?

11. Two projectiles are thrown with the same magnitude of initial velocity, one at an angle θ with respect to the level ground and the other at angle $90° - \theta$. Both projectiles will strike the ground at the same distance from the projection point. Will both projectiles be in the air for the same time interval?

12. A projectile is launched at some angle to the horizontal with some initial speed v_i, and air resistance is negligible. Is the projectile a freely falling body? What is its acceleration in the vertical direction? What is its acceleration in the horizontal direction?

13. State which of the following quantities, if any, remain constant as a projectile moves through its parabolic trajectory: (a) speed, (b) acceleration, (c) horizontal component of velocity, (d) vertical component of velocity.

14. A projectile is fired at an angle of 30° from the horizontal with some initial speed. Firing the projectile at what other angle results in the same horizontal range if the initial speed is the same in both cases? Neglect air resistance.

15. The maximum range of a projectile occurs when it is launched at an angle of 45.0° with the horizontal, if air resistance is neglected. If air resistance is not neglected, will the optimum angle be greater or less than 45.0°? Explain.

16. A projectile is launched on the Earth with some initial velocity. Another projectile is launched on the Moon with the same initial velocity. Neglecting air resistance, which projectile has the greater range? Which reaches the greater altitude? (Note that the free-fall acceleration on the Moon is about 1.6 m/s^2.)

17. A coin on a table is given an initial horizontal velocity such that it ultimately leaves the end of the table and hits the floor. At the instant the coin leaves the end of the table, a ball is released from the same height and falls to the floor. Explain why the two objects hit the floor simultaneously, even though the coin has an initial velocity.

$\boxed{18.}$ Explain whether or not the following particles have an acceleration: (a) a particle moving in a straight line with constant speed and (b) a particle moving around a curve with constant speed.

19. Correct the following statement: "The racing car rounds the turn at a constant velocity of 90 miles per hour."

20. At the end of a pendulum's arc, its velocity is zero. Is its acceleration also zero at that point?

21. An object moves in a circular path with constant speed v. (a) Is the velocity of the object constant? (b) Is its acceleration constant? Explain.

22. Describe how a driver can steer a car traveling at constant speed so that (a) the acceleration is zero or (b) the magnitude of the acceleration remains constant.

23. An ice skater is executing a figure eight, consisting of two equal, tangent circular paths. Throughout the first loop she increases her speed uniformly, and during the second loop she moves at a constant speed. Draw a motion diagram showing her velocity and acceleration vectors at several points along the path of motion.

24. Based on your observation and experience, draw a motion diagram showing the position, velocity, and acceleration vectors for a pendulum that swings in an arc carrying it from an initial position 45° to the right of the central vertical line to a final position 45° to the left of the central vertical line. The arc is a quadrant of a circle, and you should use the center of the circle as the origin for the position vectors.

25. What is the fundamental difference between the unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ and the unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$?

26. A sailor drops a wrench from the top of a sailboat's mast while the boat is moving rapidly and steadily in a straight line. Where will the wrench hit the deck? (Galileo posed this question.)

27. A ball is thrown upward in the air by a passenger on a train that is moving with constant velocity. (a) Describe the path of the ball as seen by the passenger. Describe the path as seen by an observer standing by the tracks outside the train. (b) How would these observations change if the train were accelerating along the track?

28. A passenger on a train that is moving with constant velocity drops a spoon. What is the acceleration of the spoon relative to (a) the train and (b) the Earth?

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging □ = full solution available in the *Student Solutions Manual and Study Guide*

= coached solution with hints available at http://www.pse6.com = computer useful in solving problem

= paired numerical and symbolic problems

Section 4.1 The Position, Velocity, and Acceleration Vectors

$\boxed{1.}$ A motorist drives south at 20.0 m/s for 3.00 min, then turns west and travels at 25.0 m/s for 2.00 min, and finally travels northwest at 30.0 m/s for 1.00 min. For this 6.00-min trip, find (a) the total vector displacement, (b) the average speed, and (c) the average velocity. Let the positive x axis point east.

2. A golf ball is hit off a tee at the edge of a cliff. Its x and y coordinates as functions of time are given by the following expressions:

$$x = (18.0 \text{ m/s})t$$

and $$y = (4.00 \text{ m/s})t - (4.90 \text{ m/s}^2)t^2$$

(a) Write a vector expression for the ball's position as a function of time, using the unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$. By taking derivatives, obtain expressions for (b) the velocity vector $\mathbf{v}$ as a function of time and (c) the acceleration vector $\mathbf{a}$ as a function of time. Next use unit-vector notation to write expressions for (d) the position, (e) the velocity, and (f) the acceleration of the golf ball, all at $t = 3.00$ s.

3. When the Sun is directly overhead, a hawk dives toward the ground with a constant velocity of 5.00 m/s at 60.0° below the horizontal. Calculate the speed of her shadow on the level ground.

4. The coordinates of an object moving in the xy plane vary with time according to the equations $x = -(5.00 \text{ m})\sin(\omega t)$ and $y = (4.00 \text{ m}) - (5.00 \text{ m})\cos(\omega t)$, where ω is a constant and t is in seconds. (a) Determine the components of velocity and components of acceleration at $t = 0$. (b) Write expressions for the position vector, the velocity vector, and the acceleration vector at any time $t > 0$. (c) Describe the path of the object in an xy plot.

Section 4.2 Two-Dimensional Motion with Constant Acceleration

5. At $t = 0$, a particle moving in the xy plane with constant acceleration has a velocity of $\mathbf{v}_i = (3.00\hat{\mathbf{i}} - 2.00\hat{\mathbf{j}})$ m/s and is at the origin. At $t = 3.00$ s, the particle's velocity is $\mathbf{v} = (9.00\hat{\mathbf{i}} + 7.00\hat{\mathbf{j}})$ m/s. Find (a) the acceleration of the particle and (b) its coordinates at any time t.

6. The vector position of a particle varies in time according to the expression $\mathbf{r} = (3.00\hat{\mathbf{i}} - 6.00t^2\hat{\mathbf{j}})$ m. (a) Find expressions for the velocity and acceleration as functions of time. (b) Determine the particle's position and velocity at $t = 1.00$ s.

7. A fish swimming in a horizontal plane has velocity $\mathbf{v}_i = (4.00\hat{\mathbf{i}} + 1.00\hat{\mathbf{j}})$ m/s at a point in the ocean where the position relative to a certain rock is $\mathbf{r}_i = (10.0\hat{\mathbf{i}} - 4.00\hat{\mathbf{j}})$ m. After the fish swims with constant acceleration for 20.0 s, its velocity is $\mathbf{v} = (20.0\hat{\mathbf{i}} - 5.00\hat{\mathbf{j}})$ m/s. (a) What are the components of the acceleration? (b) What is the direction of the acceleration with respect to unit vector $\hat{\mathbf{i}}$? (c) If the fish maintains constant acceleration, where is it at $t = 25.0$ s, and in what direction is it moving?

8. A particle initially located at the origin has an acceleration of $\mathbf{a} = 3.00\hat{\mathbf{j}}$ m/s^2 and an initial velocity of $\mathbf{v}_i = 500\hat{\mathbf{i}}$ m/s. Find (a) the vector position and velocity at any time t and (b) the coordinates and speed of the particle at $t = 2.00$ s.

9. It is not possible to see very small objects, such as viruses, using an ordinary light microscope. An electron microscope can view such objects using an electron beam instead of a light beam. Electron microscopy has proved invaluable for investigations of viruses, cell membranes and subcellular structures, bacterial surfaces, visual receptors, chloroplasts, and the contractile properties of muscles. The "lenses" of an electron microscope consist of electric and magnetic fields that control the electron beam. As an example of the manipulation of an electron beam, consider an electron traveling away from the origin along the x axis in the xy plane with initial velocity $\mathbf{v}_i = v_i\hat{\mathbf{i}}$. As it passes through the region $x = 0$ to $x = d$, the electron experiences acceleration $\mathbf{a} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}}$, where a_x and a_y are constants. For the case $v_i = 1.80 \times 10^7$ m/s, $a_x = 8.00 \times 10^{14}$ m/s^2 and $a_y = 1.60 \times 10^{15}$ m/s^2, determine at $x = d = 0.0100$ m (a) the position of the electron, (b) the velocity of the electron, (c) the speed of the electron, and (d) the direction of travel of the electron (i.e., the angle between its velocity and the x axis).

Section 4.3 Projectile Motion

Note: Ignore air resistance in all problems and take $g = 9.80$ m/s^2 at the Earth's surface.

10. To start an avalanche on a mountain slope, an artillery shell is fired with an initial velocity of 300 m/s at 55.0° above the horizontal. It explodes on the mountainside 42.0 s after firing. What are the x and y coordinates of the shell where it explodes, relative to its firing point?

11. In a local bar, a customer slides an empty beer mug down the counter for a refill. The bartender is momentarily distracted and does not see the mug, which slides off the counter and strikes the floor 1.40 m from the base of the counter. If the height of the counter is 0.860 m, (a) with what velocity did the mug leave the counter, and (b) what was the direction of the mug's velocity just before it hit the floor?

12. In a local bar, a customer slides an empty beer mug down the counter for a refill. The bartender is momentarily distracted and does not see the mug, which slides off the counter and strikes the floor at distance d from the base of the counter. The height of the counter is h. (a) With what velocity did the mug leave the counter, and (b) what was the direction of the mug's velocity just before it hit the floor?

13. One strategy in a snowball fight is to throw a snowball at a high angle over level ground. While your opponent is watching the first one, a second snowball is thrown at a low angle timed to arrive before or at the same time as the first one. Assume both snowballs are thrown with a speed of 25.0 m/s. The first one is thrown at an angle of 70.0° with respect to the horizontal. (a) At what angle should the second snowball be thrown to arrive at the same point as the first? (b) How many seconds later should the second snowball be thrown after the first to arrive at the same time?

14. An astronaut on a strange planet finds that she can jump a maximum horizontal distance of 15.0 m if her initial speed is 3.00 m/s. What is the free-fall acceleration on the planet?

15. A projectile is fired in such a way that its horizontal range is equal to three times its maximum height. What is the angle of projection?

16. A rock is thrown upward from the level ground in such a way that the maximum height of its flight is equal to its horizontal range d. (a) At what angle θ is the rock thrown? (b) **What If?** Would your answer to part (a) be different on a different planet? (c) What is the range d_{max} the rock can attain if it is launched at the same speed but at the optimal angle for maximum range?

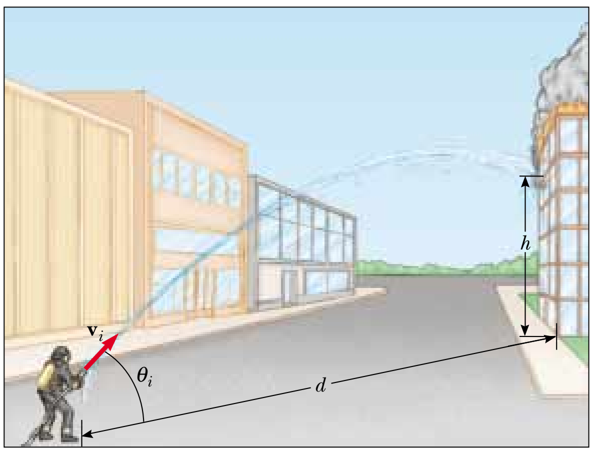
17. A ball is tossed from an upper-story window of a building. The ball is given an initial velocity of 8.00 m/s at an angle of 20.0° below the horizontal. It strikes the ground 3.00 s later. (a) How far horizontally from the base of the building does the ball strike the ground? (b) Find the height from which the ball was thrown. (c) How long does it take the ball to reach a point 10.0 m below the level of launching?

18. The small archerfish (length 20 to 25 cm) lives in brackish waters of southeast Asia from India to the Philippines. This aptly named creature captures its prey by shooting a stream of water drops at an insect, either flying or at rest. The bug falls into the water and the fish gobbles it up. The archerfish has high accuracy at distances of 1.2 m to 1.5 m, and it sometimes makes hits at distances up to 3.5 m. A groove in the roof of its mouth, along with a curled tongue, forms a tube that enables the fish to impart high velocity to the water in its mouth when it suddenly closes its gill flaps. Suppose the archerfish shoots at a target

2.00 m away, at an angle of 30.0° above the horizontal. With what velocity must the water stream be launched if it is not to drop more than 3.00 cm vertically on its path to the target?

19. A place-kicker must kick a football from a point 36.0 m (about 40 yards) from the goal, and half the crowd hopes the ball will clear the crossbar, which is 3.05 m high. When kicked, the ball leaves the ground with a speed of 20.0 m/s at an angle of 53.0° to the horizontal. (a) By how much does the ball clear or fall short of clearing the crossbar? (b) Does the ball approach the crossbar while still rising or while falling?

20. A firefighter, a distance d from a burning building, directs a stream of water from a fire hose at angle θ_i above the horizontal as in Figure P4.20. If the initial speed of the stream is v_i, at what height h does the water strike the building?



Frederick McKinney/Getty Images

Figure P4.20

21. A playground is on the flat roof of a city school, 6.00 m above the street below. The vertical wall of the building is 7.00 m high, to form a meter-high railing around the playground. A ball has fallen to the street below, and a passerby returns it by launching it at an angle of 53.0° above the horizontal at a point 24.0 meters from the base of the building wall. The ball takes 2.20 s to reach a point vertically above the wall. (a) Find the speed at which the ball was launched. (b) Find the vertical distance by which the ball clears the wall. (c) Find the distance from the wall to the point on the roof where the ball lands.

22. A dive bomber has a velocity of 280 m/s at an angle θ below the horizontal. When the altitude of the aircraft is 2.15 km, it releases a bomb, which subsequently hits a target on the ground. The magnitude of the displacement from the point of release of the bomb to the target is 3.25 km. Find the angle θ.

23. A soccer player kicks a rock horizontally off a 40.0-m high cliff into a pool of water. If the player hears the sound of the splash 3.00 s later, what was the initial speed given to the rock? Assume the speed of sound in air to be 343 m/s.

24. A basketball star covers 2.80 m horizontally in a jump to dunk the ball (Fig. P4.24). His motion through space can be modeled precisely as that of a particle at his *center of mass,* which we will define in Chapter 9. His center of mass is at elevation 1.02 m when he leaves the floor. It reaches a maximum height of 1.85 m above the floor, and is at elevation 0.900 m when he touches down again. Determine (a) his time of

Jed Jacobsohn/Allsport/Getty Images

Bill Lee/Dembinsky Photo Associates

Figure P4.24

flight (his "hang time"), (b) his horizontal and (c) vertical velocity components at the instant of takeoff, and (d) his takeoff angle. (e) For comparison, determine the hang time of a whitetail deer making a jump with center-of-mass elevations $y_i = 1.20$ m, $y_{max} = 2.50$ m, $y_f = 0.700$ m.

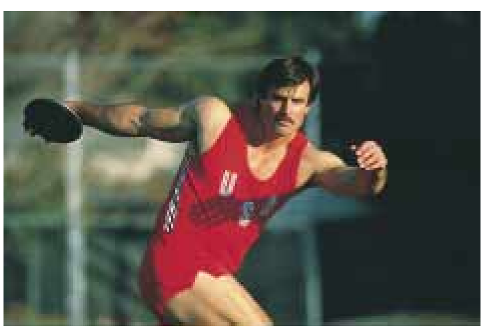
25. An archer shoots an arrow with a velocity of 45.0 m/s at an angle of 50.0° with the horizontal. An assistant standing on the level ground 150 m downrange from the launch point throws an apple straight up with the minimum initial speed necessary to meet the path of the arrow. (a) What is the initial speed of the apple? (b) At what time after the arrow launch should the apple be thrown so that the arrow hits the apple?

26. A fireworks rocket explodes at height h, the peak of its vertical trajectory. It throws out burning fragments in all directions, but all at the same speed v. Pellets of solidified metal fall to the ground without air resistance. Find the smallest angle that the final velocity of an impacting fragment makes with the horizontal.

Section 4.4 Uniform Circular Motion

Note: Problems 8, 10, 12, and 16 in Chapter 6 can also be assigned with this section.

27. The athlete shown in Figure P4.27 rotates a 1.00-kg discus along a circular path of radius 1.06 m. The maximum speed of the discus is 20.0 m/s. Determine the magnitude of the maximum radial acceleration of the discus.



Sam Sargent/Liaison International

Figure P4.27

28. From information on the endsheets of this book, compute the radial acceleration of a point on the surface of the Earth at the equator, due to the rotation of the Earth about its axis.

29. A tire 0.500 m in radius rotates at a constant rate of 200 rev/min. Find the speed and acceleration of a small stone lodged in the tread of the tire (on its outer edge).

30. As their booster rockets separate, Space Shuttle astronauts typically feel accelerations up to $3g$, where $g = 9.80$ m/s^2. In their training, astronauts ride in a device where they experience such an acceleration as a centripetal acceleration. Specifically, the astronaut is fastened securely at the end of a mechanical arm that then turns at constant speed in a horizontal circle. Determine the rotation rate, in revolutions per second, required to give an astronaut a centripetal acceleration of $3.00g$ while in circular motion with radius 9.45 m.

31. Young David who slew Goliath experimented with slings before tackling the giant. He found that he could revolve a sling of length 0.600 m at the rate of 8.00 rev/s. If he increased the length to 0.900 m, he could revolve the sling only 6.00 times per second. (a) Which rate of rotation gives the greater speed for the stone at the end of the sling? (b) What is the centripetal acceleration of the stone at 8.00 rev/s? (c) What is the centripetal acceleration at 6.00 rev/s?

32. The astronaut orbiting the Earth in Figure P4.32 is preparing to dock with a Westar VI satellite. The satellite is in a circular orbit 600 km above the Earth's surface, where the free-fall acceleration is 8.21 m/s^2. Take the radius of the Earth as 6 400 km. Determine the speed of the satellite and the time interval required to complete one orbit around the Earth.

Courtesy of NASA

Figure P4.32

Section 4.5 Tangential and Radial Acceleration

33. A train slows down as it rounds a sharp horizontal turn, slowing from 90.0 km/h to 50.0 km/h in the 15.0 s that it takes to round the bend. The radius of the curve is 150 m. Compute the acceleration at the moment the train speed reaches 50.0 km/h. Assume it continues to slow down at this time at the same rate.

34. An automobile whose speed is increasing at a rate of 0.600 m/s^2 travels along a circular road of radius 20.0 m. When the instantaneous speed of the automobile is 4.00 m/s, find (a) the tangential acceleration component, (b) the centripetal acceleration component, and (c) the magnitude and direction of the total acceleration.

35. Figure P4.35 represents the total acceleration of a particle moving clockwise in a circle of radius 2.50 m at a certain instant of time. At this instant, find (a) the radial acceleration, (b) the speed of the particle, and (c) its tangential acceleration.

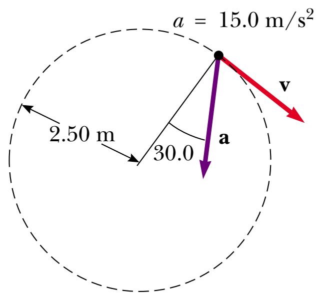


Figure P4.35

36. A ball swings in a vertical circle at the end of a rope 1.50 m long. When the ball is 36.9° past the lowest point on its way up, its total acceleration is $(-22.5\hat{\mathbf{i}} + 20.2\hat{\mathbf{j}})$ m/s². At that instant, (a) sketch a vector diagram showing the components of its acceleration, (b) determine the magnitude of its radial acceleration, and (c) determine the speed and velocity of the ball.

37. A race car starts from rest on a circular track. The car increases its speed at a constant rate a_t as it goes once around the track. Find the angle that the total acceleration of the car makes—with the radius connecting the center of the track and the car—at the moment the car completes the circle.

Section 4.6 Relative Velocity and Relative Acceleration

38. Heather in her Corvette accelerates at the rate of $(3.00\hat{\mathbf{i}} - 2.00\hat{\mathbf{j}})$ m/s², while Jill in her Jaguar accelerates at $(1.00\hat{\mathbf{i}} + 3.00\hat{\mathbf{j}})$ m/s². They both start from rest at the origin of an xy coordinate system. After 5.00 s, (a) what is Heather's speed with respect to Jill, (b) how far apart are they, and (c) what is Heather's acceleration relative to Jill?

39. A car travels due east with a speed of 50.0 km/h. Raindrops are falling at a constant speed vertically with respect to the Earth. The traces of the rain on the side windows of the car make an angle of 60.0° with the vertical. Find the velocity of the rain with respect to (a) the car and (b) the Earth.

40. How long does it take an automobile traveling in the left lane at 60.0 km/h to pull alongside a car traveling in the same direction in the right lane at 40.0 km/h if the cars' front bumpers are initially 100 m apart?

41. A river has a steady speed of 0.500 m/s. A student swims upstream a distance of 1.00 km and swims back to the starting point. If the student can swim at a speed of 1.20 m/s in still water, how long does the trip take? Compare this with the time the trip would take if the water were still.

42. The pilot of an airplane notes that the compass indicates a heading due west. The airplane's speed relative to the air is 150 km/h. If there is a wind of 30.0 km/h toward the north, find the velocity of the airplane relative to the ground.

43. Two swimmers, Alan and Beth, start together at the same point on the bank of a wide stream that flows with a speed v. Both move at the same speed $c (c > v)$, relative to the water. Alan swims downstream a distance L and then upstream the same distance. Beth swims so that her motion relative to the Earth is perpendicular to the banks of the stream. She swims the distance L and then back the same distance, so that both swimmers return to the starting point. Which swimmer returns first? (*Note:* First guess the answer.)

44. A bolt drops from the ceiling of a train car that is accelerating northward at a rate of 2.50 m/s². What is the acceleration of the bolt relative to (a) the train car? (b) the Earth?

45. A science student is riding on a flatcar of a train traveling along a straight horizontal track at a constant speed of 10.0 m/s. The student throws a ball into the air along a path that he judges to make an initial angle of 60.0° with the horizontal and to be in line with the track. The student's professor, who is standing on the ground nearby, observes the ball to rise vertically. How high does she see the ball rise?

46. A Coast Guard cutter detects an unidentified ship at a distance of 20.0 km in the direction 15.0° east of north. The ship is traveling at 26.0 km/h on a course at 40.0° east of north. The Coast Guard wishes to send a speedboat to intercept the vessel and investigate it. If the speedboat travels 50.0 km/h, in what direction should it head? Express the direction as a compass bearing with respect to due north.

Additional Problems

47. *The "Vomit Comet."* In zero-gravity astronaut training and equipment testing, NASA flies a KC135A aircraft along a parabolic flight path. As shown in Figure P4.47, the aircraft climbs from 24 000 ft to 31 000 ft, where it enters the zero-g parabola with a velocity of 143 m/s nose-high at 45.0° and exits with velocity 143 m/s at 45.0° nose-low. During this portion of the flight the aircraft and objects inside its padded cabin are in free fall—they have gone ballistic. The aircraft then pulls out of the dive with an upward acceleration of 0.800g, moving in a vertical circle with radius 4.13 km. (During this portion of the flight, occupants of the plane perceive an acceleration of 1.8g.) What are the aircraft (a) speed and (b) altitude at the top of the maneuver? (c) What is the time spent in zero gravity? (d) What is the speed of the aircraft at the bottom of the flight path?

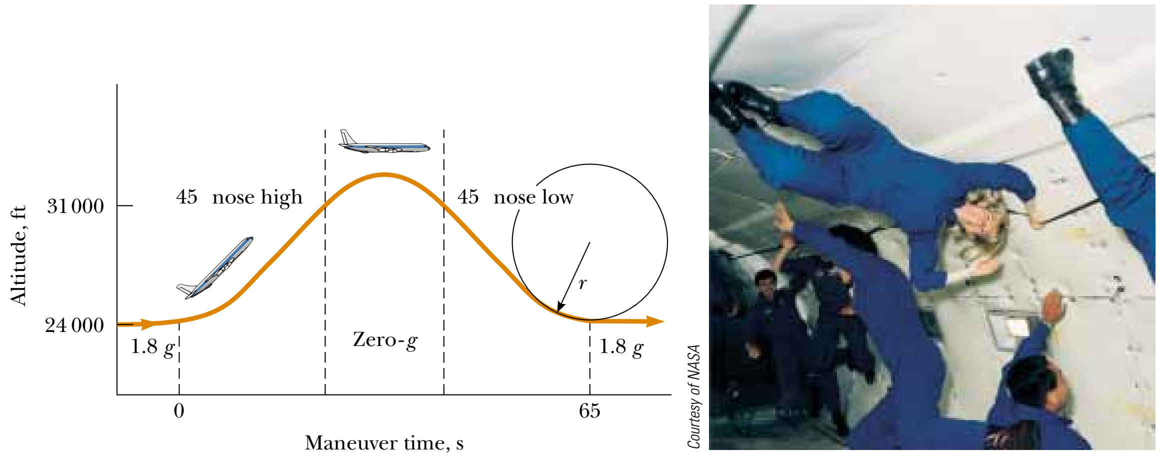


Figure P4.47

48. As some molten metal splashes, one droplet flies off to the east with initial velocity v_i at angle θ_i above the horizontal, and another droplet to the west with the same speed at the same angle above the horizontal, as in Figure P4.48. In terms of v_i and θ_i, find the distance between them as a function of time.

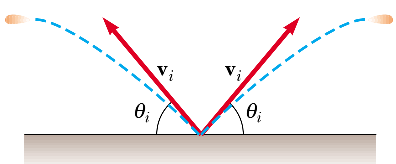


Figure P4.48

49. A ball on the end of a string is whirled around in a horizontal circle of radius 0.300 m. The plane of the circle is 1.20 m above the ground. The string breaks and the ball lands 2.00 m (horizontally) away from the point on the ground directly beneath the ball's location when the string breaks. Find the radial acceleration of the ball during its circular motion.

50. A projectile is fired up an incline (incline angle ϕ) with an initial speed v_i at an angle θ_i with respect to the horizontal ($\theta_i > \phi$), as shown in Figure P4.50. (a) Show that the projectile travels a distance d up the incline, where

$$d = \frac{2v_i^2 \cos\theta_i \sin(\theta_i - \phi)}{g\cos^2\phi}$$

(b) For what value of θ_i is d a maximum, and what is that maximum value?

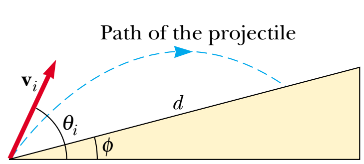


Figure P4.50

51. Barry Bonds hits a home run so that the baseball just clears the top row of bleachers, 21.0 m high, located 130 m from home plate. The ball is hit at an angle of 35.0° to the horizontal, and air resistance is negligible. Find (a) the initial speed of the ball, (b) the time at which the ball reaches the cheap seats, and (c) the velocity components and the speed of the ball when it passes over the top row. Assume the ball is hit at a height of 1.00 m above the ground.

52. An astronaut on the surface of the Moon fires a cannon to launch an experiment package, which leaves the barrel moving horizontally. (a) What must be the muzzle speed of the package so that it travels completely around the Moon and returns to its original location? (b) How long does this trip around the Moon take? Assume that the free-fall acceleration on the Moon is one-sixth that on the Earth.

53. A pendulum with a cord of length $r = 1.00$ m swings in a vertical plane (Fig. P4.53). When the pendulum is in the two horizontal positions $\theta = 90.0°$ and $\theta = 270°$, its speed is 5.00 m/s. (a) Find the magnitude of the radial acceleration and tangential acceleration for these positions. (b) Draw vector diagrams to determine the direc-

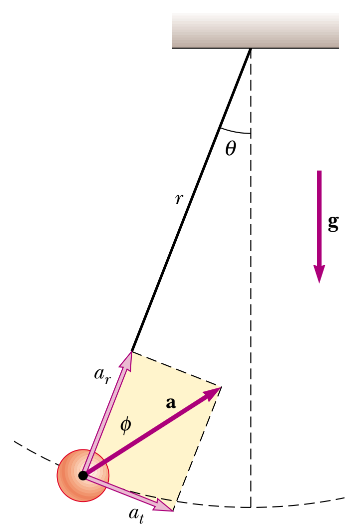


Figure P4.53

tion of the total acceleration for these two positions. (c) Calculate the magnitude and direction of the total acceleration.

54. A basketball player who is 2.00 m tall is standing on the floor 10.0 m from the basket, as in Figure P4.54. If he shoots the ball at a 40.0° angle with the horizontal, at what initial speed must he throw so that it goes through the hoop without striking the backboard? The basket height is 3.05 m.

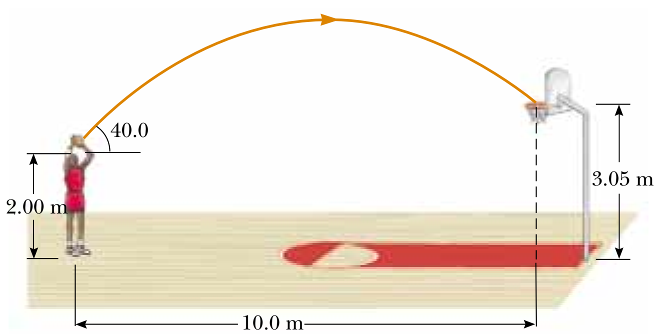


Figure P4.54

55. When baseball players throw the ball in from the outfield, they usually allow it to take one bounce before it reaches the infield, on the theory that the ball arrives sooner that way. Suppose that the angle at which a bounced ball leaves the ground is the same as the angle at which the outfielder threw it, as in Figure P4.55, but that the ball's speed after the bounce is one half of what it was before the bounce. (a) Assuming the ball is always thrown with the same initial speed, at what angle θ should the fielder throw the ball to make it go the same distance D with one bounce (blue path) as a ball thrown upward at 45.0° with no bounce (green path)? (b) Determine the ratio of the times for the one-bounce and no-bounce throws.

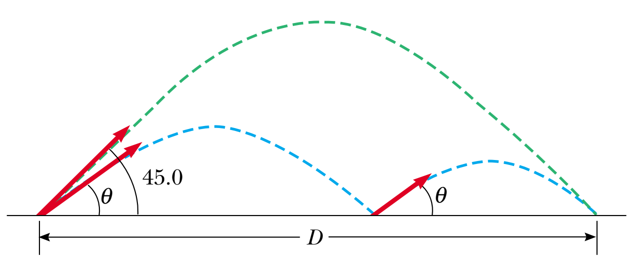


Figure P4.55

56. A boy can throw a ball a maximum horizontal distance of R on a level field. How far can he throw the same ball vertically upward? Assume that his muscles give the ball the same speed in each case.

57. A stone at the end of a sling is whirled in a vertical circle of radius 1.20 m at a constant speed $v_0 = 1.50$ m/s as in Figure P4.57. The center of the sling is 1.50 m above the ground. What is the range of the stone if it is released when the sling is inclined at 30.0° with the horizontal (a) at Ⓐ? (b) at Ⓑ? What is the acceleration of the stone (c) just before it is released at Ⓐ? (d) just after it is released at Ⓐ?

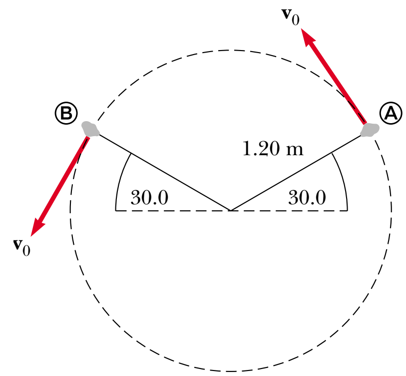


Figure P4.57

58. A quarterback throws a football straight toward a receiver with an initial speed of 20.0 m/s, at an angle of 30.0° above the horizontal. At that instant, the receiver is 20.0 m from the quarterback. In what direction and with what constant speed should the receiver run in order to catch the football at the level at which it was thrown?

59. Your grandfather is copilot of a bomber, flying horizontally over level terrain, with a speed of 275 m/s relative to the ground, at an altitude of 3 000 m. (a) The bombardier releases one bomb. How far will it travel horizontally between its release and its impact on the ground? Neglect the effects of air resistance. (b) Firing from the people on the ground suddenly incapacitates the bombardier before he can call, "Bombs away!" Consequently, the pilot maintains the plane's original course, altitude, and speed through a storm of flak. Where will the plane be when the bomb hits the ground? (c) The plane has a telescopic bomb sight set so that the bomb hits the target seen in the sight at the time of release. At what angle from the vertical was the bomb sight set?

60. A high-powered rifle fires a bullet with a muzzle speed of 1.00 km/s. The gun is pointed horizontally at a large bull's eye target—a set of concentric rings—200 m away. (a) How far below the extended axis of the rifle barrel does a bullet hit the target? The rifle is equipped with a telescopic sight. It is "sighted in" by adjusting the axis of the telescope so that it points precisely at the location where the bullet hits the target at 200 m. (b) Find the angle between the telescope axis and the rifle barrel axis. When shooting at a target at a distance other than 200 m, the marksman uses the telescopic sight, placing its crosshairs to "aim high" or "aim low" to compensate for the different range. Should she aim high or low, and approximately how far from the bull's eye, when the target is at a distance of (c) 50.0 m, (d) 150 m, or (e) 250 m? *Note:* The trajectory of the bullet is everywhere so nearly horizontal that it is a good approximation to model the bullet as fired horizontally in each case. **What if** the target is uphill or downhill? (f) Suppose the target is 200 m away, but the sight line to the target is above the horizontal by 30°. Should the marksman aim high, low, or right on? (g) Suppose the target is downhill by 30°. Should the marksman aim high, low, or right on? Explain your answers.

61. A hawk is flying horizontally at 10.0 m/s in a straight line, 200 m above the ground. A mouse it has been carrying struggles free from its grasp. The hawk continues on its path at the same speed for 2.00 seconds before attempting to retrieve its prey. To accomplish the retrieval, it dives in a straight line at constant speed and recaptures the mouse 3.00 m above the ground. (a) Assuming no air resistance, find the diving speed of the hawk. (b) What angle did the hawk make with the horizontal during its descent? (c) For how long did the mouse "enjoy" free fall?

62. A person standing at the top of a hemispherical rock of radius R kicks a ball (initially at rest on the top of the rock) to give it horizontal velocity $\mathbf{v}_i$ as in Figure P4.62. (a) What must be its minimum initial speed if the ball is never to hit the rock after it is kicked? (b) With this initial speed, how far from the base of the rock does the ball hit the ground?

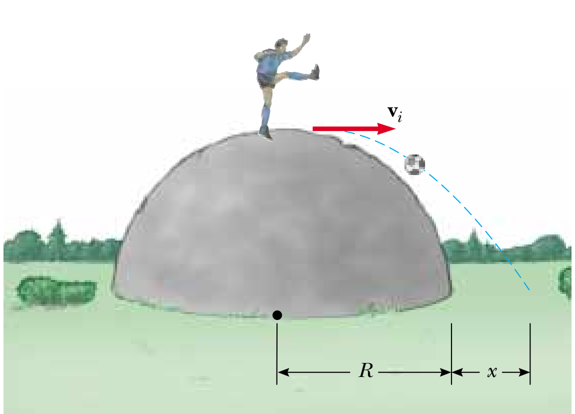


Figure P4.62

63. A car is parked on a steep incline overlooking the ocean, where the incline makes an angle of 37.0° below the horizontal. The negligent driver leaves the car in neutral, and the parking brakes are defective. Starting from rest at $t = 0$, the car rolls down the incline with a constant acceleration of 4.00 m/s^2, traveling 50.0 m to the edge of a vertical cliff. The cliff is 30.0 m above the ocean. Find (a) the speed of the car when it reaches the edge of the cliff and the time at which it arrives there, (b) the velocity of the car when it lands in the ocean, (c) the total time interval that the car is in motion, and (d) the position of the car when it lands in the ocean, relative to the base of the cliff.

64. A truck loaded with cannonball watermelons stops suddenly to avoid running over the edge of a washed-out bridge (Fig. P4.64). The quick stop causes a number of melons to fly off the truck. One melon rolls over the edge with an initial speed $v_i = 10.0$ m/s in the horizontal direction. A cross-section of the bank has the shape of the bottom half of a parabola with its vertex at the edge of the road, and with the equation $y^2 = 16x$, where x and y are measured in meters. What are the x and y coordinates of the melon when it splatters on the bank?

65. The determined coyote is out once more in pursuit of the elusive roadrunner. The coyote wears a pair of Acme jet-powered roller skates, which provide a constant horizontal acceleration of 15.0 m/s^2 (Fig. P4.65). The coyote starts at rest 70.0 m from the brink of a cliff at the instant the roadrunner zips past him in the direction of the cliff. (a) If the roadrunner moves with constant speed, determine the minimum speed he must have in order to reach the cliff before the coyote. At the edge of the cliff, the roadrunner escapes by making a sudden turn, while the coyote continues straight ahead. His skates remain horizontal and continue to operate while he is in flight, so that the coyote's acceleration while in the air is $(15.0\hat{\mathbf{i}} - 9.80\hat{\mathbf{j}})$ m/s^2. (b) If the cliff is 100 m above the flat floor of a canyon, determine where the coyote lands in the canyon. (c) Determine the components of the coyote's impact velocity.

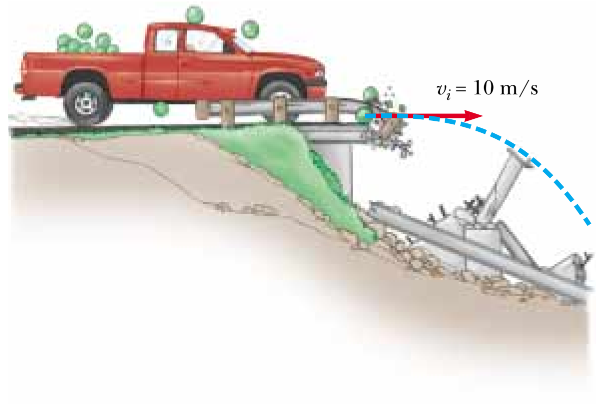


Figure P4.64

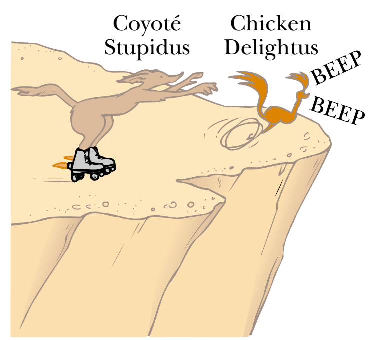


Figure P4.65

66. Do not hurt yourself; do not strike your hand against anything. Within these limitations, describe what you do to give your hand a large acceleration. Compute an order-of-magnitude estimate of this acceleration, stating the quantities you measure or estimate and their values.

67. A skier leaves the ramp of a ski jump with a velocity of 10.0 m/s, 15.0° above the horizontal, as in Figure P4.67. The slope is inclined at 50.0°, and air resistance is negligible. Find (a) the distance from the ramp to where the jumper lands and (b) the velocity components just before the landing. (How do you think the results might be affected if air resistance were included? Note that jumpers lean forward in the shape of an airfoil, with their hands at their sides, to increase their distance. Why does this work?)

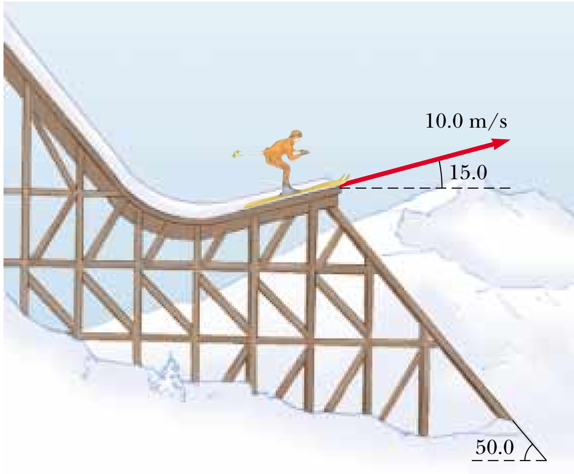


Figure P4.67

68. In a television picture tube (a cathode ray tube) electrons are emitted with velocity $\mathbf{v}_i$ from a source at the origin of coordinates. The initial velocities of different electrons make different angles θ with the x axis. As they move a distance D along the x axis, the electrons are acted on by a constant electric field, giving each a constant acceleration $\mathbf{a}$ in the x direction. At $x = D$ the electrons pass through a circular aperture, oriented perpendicular to the x axis. At the aperture, the velocity imparted to the electrons by the electric field is much larger than $\mathbf{v}_i$ in magnitude. Show that velocities of the electrons going through the aperture radiate from a certain point on the x axis, which is not the origin. Determine the location of this point. This point is called a *virtual source,* and it is important in determining where the electron beam hits the screen of the tube.

69. A fisherman sets out upstream from Metaline Falls on the Pend Oreille River in northwestern Washington State. His small boat, powered by an outboard motor, travels at a constant speed v in still water. The water flows at a lower constant speed v_w. He has traveled upstream for 2.00 km when his ice chest falls out of the boat. He notices that the chest is missing only after he has gone upstream for another 15.0 minutes. At that point he turns around and heads back downstream, all the time traveling at the same speed relative to the water. He catches up with the floating ice chest just as it is about to go over the falls at his starting point. How fast is the river flowing? Solve this problem in two ways. (a) First, use the Earth as a reference frame. With respect to the Earth, the boat travels upstream at speed $v - v_w$ and downstream at $v + v_w$. (b) A second much simpler and more elegant solution is obtained by using the water as the reference frame. This approach has important applications in many more complicated problems; examples are calculating the motion of rockets and satellites and analyzing the scattering of subatomic particles from massive targets.

70. The water in a river flows uniformly at a constant speed of 2.50 m/s between parallel banks 80.0 m apart. You are to deliver a package directly across the river, but you can swim only at 1.50 m/s. (a) If you choose to minimize the time you spend in the water, in what direction should you head? (b) How far downstream will you be carried? (c) **What If?** If you choose to minimize the distance downstream that the river carries you, in what direction should you head? (d) How far downstream will you be carried?

71. An enemy ship is on the east side of a mountain island, as shown in Figure P4.71. The enemy ship has maneuvered to within 2 500 m of the 1 800-m-high mountain peak and can shoot projectiles with an initial speed of 250 m/s. If the western shoreline is horizontally 300 m from the peak, what are the distances from the western shore at which a ship can be safe from the bombardment of the enemy ship?

72. In the **What If?** section of Example 4.7, it was claimed that the maximum range of a ski-jumper occurs for a launch angle θ given by

$$\theta = 45° - \frac{\phi}{2}$$

where ϕ is the angle that the hill makes with the horizontal in Figure 4.16. Prove this claim by deriving the equation above.

Answers to Quick Quizzes

4.1 (b). An object moving with constant velocity has $\Delta\mathbf{v} = 0$, so, according to the definition of acceleration, $\mathbf{a} = \Delta\mathbf{v}/\Delta t = 0$. Choice (a) is not correct because a particle can move at a constant speed and change direction. This possibility also makes (c) an incorrect choice.

4.2 (a). Because acceleration occurs whenever the velocity changes in any way—with an increase or decrease in

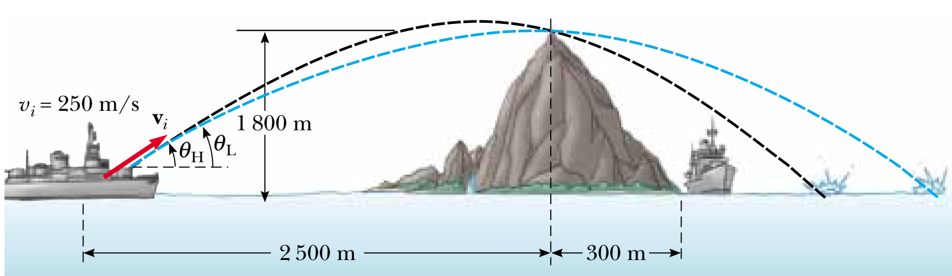


Figure P4.71

speed, a change in direction, or both—all three controls are accelerators. The gas pedal causes the car to speed up; the brake pedal causes the car to slow down. The steering wheel changes the direction of the velocity vector.

4.3 (a). You should simply throw it straight up in the air. Because the ball is moving along with you, it will follow a parabolic trajectory with a horizontal component of velocity that is the same as yours.

4.4 (b). At only one point—the peak of the trajectory—are the velocity and acceleration vectors perpendicular to each other. The velocity vector is horizontal at that point and the acceleration vector is downward.

4.5 (a). The acceleration vector is always directed downward. The velocity vector is never vertical if the object follows a path such as that in Figure 4.8.

4.6 15°, 30°, 45°, 60°, 75°. The greater the maximum height, the longer it takes the projectile to reach that altitude and then fall back down from it. So, as the launch angle increases, the time of flight increases.

4.7 (c). We cannot choose (a) or (b) because the centripetal acceleration vector is not constant—it continuously changes in direction. Of the remaining choices, only (c) gives the correct perpendicular relationship between $\mathbf{a}_c$ and $\mathbf{v}$.

4.8 (d). Because the centripetal acceleration is proportional to the square of the speed, doubling the speed increases the acceleration by a factor of 4.

4.9 (b). The velocity vector is tangent to the path. If the acceleration vector is to be parallel to the velocity vector, it must also be tangent to the path. This requires that the acceleration vector have no component perpendicular to the path. If the path were to change direction, the acceleration vector would have a radial component, perpendicular to the path. Thus, the path must remain straight.

4.10 (d). The velocity vector is tangent to the path. If the acceleration vector is to be perpendicular to the velocity vector, it must have no component tangent to the path. On the other hand, if the speed is changing, there *must* be a component of the acceleration tangent to the path. Thus, the velocity and acceleration vectors are never perpendicular in this situation. They can only be perpendicular if there is no change in the speed.

4.11 (c). Passenger A sees the coffee pouring in a "normal" parabolic path, just as if she were standing on the ground pouring it. The stationary observer B sees the coffee moving in a parabolic path that is extended horizontally due to the constant horizontal velocity of 60 mi/h.